SKOLIAD No. 104

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Please send your solutions to the problems in this edition by ***March 1, 2008***. A copy of **MATHEMATICAL MAYHEM Vol. 6** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Le concours de ce mois-ci proviennent du 23ième Concours W.J. Blundon de Mathématiques. Je remercie Don Rideout de l'Université Mémorial de Terre-Neuve qui a eu l'amabilité de me fournir les questions.

23ième Concours W.J. Blundon de Mathématiques
parrainé par la Société Mathématique du Canada et le département de mathématique et de statistique de l'Université Mémorial de Terre-Neuve
22 Février 2006

1. Si $\log_a x = \log_b y$, montrer que chacun d'eux égale $\log_{ab} xy$.

2. De combien de manières peut-on changer un billet de $20$$ en 25 sous et 10 sous, si on utilise au moins un de chaque sou ?

3. Si une femme quitte lors d'une fête, alors 20% des gens restant sont des femmes. Si, à la place, une femme se joint à la fête, alors 25% des gens restant sont des femmes. Combien d'hommes se trouvent à la fête ?

4. Trouver deux facteurs de $2^{48} - 1$ qui se trouvent entre 60 et 70.

5. Les changements annuels de population dans une ville dans quatre années consécutives sont respectivement 25% de plus, 25% de plus, 25% de moins et 25% de moins. Trouvez le pourcentage de changement net sur les quatre ans au pourcent le plus près.

6. Si $x + y = 5$ et $xy = 1$, trouver $x^3 + y^3$.

7. Le point $(4, 1)$ est sur la droite passant par $(4, 1)$ et perpendiculaire à la droite $y = 2x + 1$. Trouvez l'aire du triangle formé par la droite $y = 2x + 1$, sa perpendiculaire et l'axe des x.

8. Un point est choisi au hasard à l'intérieur d'un triangle équilatéral. À partir de ce point, on trace les trois perpendiculaires aux côtés. Montrez que la somme de ces trois segments est de même longueur qu'une hauteur du triangle.

9. Trouver tous les triplets d'entiers positifs (x, y, z) satifaisant aux équations

$$x^2 + y - z = 100 \quad \text{et} \quad x + y^2 - z = 124\,.$$

10. Combien de racines a l'équation $\sin x = \frac{1}{100}\,x$?

23rd W.J. Blundon Mathematics Contest
Sponsored by the Canadian Mathematical Society and lThe Department of Mathematics and Statistics Memorial University of Newfoundland
February 22, 2006

1. If $\log_a x = \log_b y$, show that each is also equal to $\log_{ab} xy$.

2. In how many ways can 20 dollars be changed into dimes and quarters, with at least one of each coin used?

3. If one of the women at a party leaves, then 20% of the people remaining at the party are women. If, instead, another woman arrives at the party, then 25% of the people at the party are women. How many men are at the party?

4. Find two factors of $2^{48} - 1$ between 60 and 70.

5. The yearly changes in the population census of a town for four consecutive years are, respectively, 25% increase, 25% increase, 25% decrease, and 25% decrease. Find the net percent change to the nearest percent over the four years.

6. If $x + y = 5$ and $xy = 1$, find $x^3 + y^3$.

7. The point $(4, 1)$ is on the line that passes through the point $(4, 1)$ and is perpendicular to the line $y = 2x + 1$. Find the area of the triangle formed by the line $y = 2x + 1$, the given perpendicular line, and the x-axis.

8. An arbitrary point is selected inside an equilateral triangle. From this point perpendiculars are dropped to each side of the triangle. Show that the sum of the lengths of these perpendiculars is equal to the length of the altitude of the triangle.

9. Find all positive integer triples (x, y, z) satisfying the equations

$$x^2 + y - z = 100 \quad \text{and} \quad x + y^2 - z = 124\,.$$

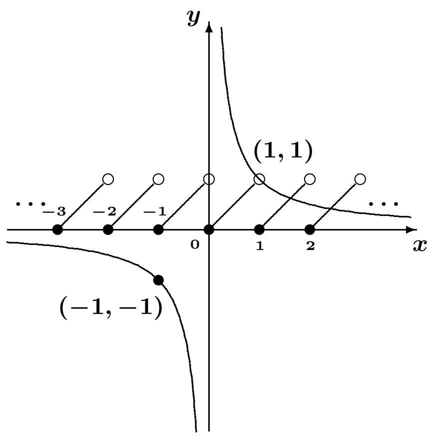
10. How many roots are there to the equation $\sin x = \frac{1}{100}\,x$?

Next we give the solutions to the Concours Montmorency 2004–2005 run by Collège Montmorency [2007 : 3–5].

1. The golden ratio $N = \frac{1+\sqrt{5}}{2} \approx 1.618033989\ldots$ has the remarkable property that its multiplicative inverse $1/N$ is equal to its decimal part $0.618033989\ldots$. Find another number with this property.

Two interesting solutions were presented with a nice generalization. We first present the solution by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

A non-zero real number x has this remarkable property if and only if $x - \lfloor x \rfloor = 1/x$. Thus, the set X of all such non-zero real numbers is the set of x-coordinates of the points of intersection of the graphs of $y = x - \lfloor x \rfloor$ and $y = 1/x$, as illustrated.



Clearly, no such non-zero real numbers are negative. Furthermore, all the positive solutions are given by $\frac{1}{2}\left(k + \sqrt{k^2+4}\right)$ for positive integers k, as can be seen by solving for the intersection of the graphs of $y = x - k$ and $y = 1/x$.

Next we feature the solution by Justin Yang, student, Lord Bing Secondary School, Vancouver, BC.

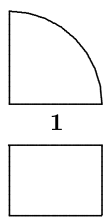
Let N be a number having the remarkable property in the problem. Hence, $N - \lfloor N \rfloor = 1/N$. When $\lfloor N \rfloor = n$, we have $N - n = 1/N$. Using the quadratic formula, we get

$$N = \frac{n + \sqrt{n^2+4}}{2}\,.$$

If $n = 1$, we get the given solution $N = \frac{1+\sqrt{5}}{2}$, the golden ratio. If $n = 2$, we get $N = 1 + \sqrt{2}$, a new solution, as required.

Also solved by JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; and VEDULA N. MURTY, Dover, PA, USA. There was one incorrect solution submitted.

2. Consider a quarter circle of radius 1.



(a) Find a rectangle having the same area and the same perimeter as the quarter circle.

(b) For a complete circle of radius 1, is it possible to find such a rectangle, having an area and a perimeter equal to that of the circle? Justify your answer.

Solution to part (a) *by Natalia Desy, student, Palembang, Indonesia.*

Let the rectangle have sides of length x and y. Then the area is $xy = \frac{\pi}{4}$ and the perimeter is $2x + 2y = 2 + \frac{\pi}{2}$. Substituting $y = \pi/(4x)$ into the second equation, we get $4x^2 - (4 + \pi)x + \pi = 0$. Using the quadratic formula, we get $x = \dfrac{4 + \pi \pm (4 - \pi)}{8}$. We may assign either value to x; the other value will be y. Thus, our rectangle has sides 1 and $\frac{\pi}{4}$.

Solution to part (b) *by Vedula N. Murty, Dover, PA, USA.*

It is not possible to find such a rectangle. If the rectangle has to have the same perimeter and area as a circle with radius 1, we have $x + y = \pi$ and $xy = \pi$. Since $(x + y)^2 \geq 4xy$ for all real numbers x and y, we then have $\pi^2 \geq 4\pi$, implying that $\pi \geq 4$, which is false. Therefore, no such real numbers x and y exist.

Also solved by JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB; and JUSTIN YANG, student, Lord Bing Secondary School, Vancouver, BC. Desy and Murty solved both parts (a) *and* (b).

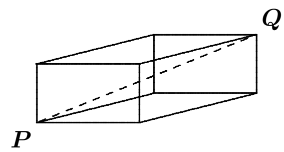
3. A barrel is filled with water. We empty half of its contents and then add a litre of water. After doing this operation seven consecutive times, we are left with three litres of water in the barrel. How many litres were in the barrel at the beginning?

Solution by Natalia Desy, student, Palembang, Indonesia.

Let x be the volume of the barrel, in litres. If we empty half, we are left with $\frac{1}{2}x$, and adding a litre gives $\frac{1}{2}x + 1$. After repeating these operations 7 times, the barrel contains $\frac{1}{128}x + \frac{127}{64}$ liters of water. Setting this equal to 3 and solving, we get $x = 130$. Hence, the initial amount of water in the barrel is 130 litres.

Also solved by JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; and MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB.

4. The areas of three faces of a rectangular parallelepiped are 18 cm^2, 40 cm^2 and 80 cm^2. Find: (a) its volume; (b) the length of its diagonal $\overline{PQ}$.



Solution by Justin Yang, student, Lord Bing Secondary School, Vancouver, BC.

(a) Assume the rectangular parallelepiped has side lengths a, b, and c $(0 < a < b < c)$. Hence, we have $ab = 18$, $bc = 80$, and $ac = 40$. Multiplying the three equations, we obtain $a^2b^2c^2 = 57600$. The volume is $abc = \sqrt{57600} = 240\ \text{cm}^3$.

(b) We have $a = abc/(bc) = 240/80 = 3$; similarly, we get $b = 6$ and $c = 40/3$. Hence, $PQ = \sqrt{3^2 + 6^2 + (40/3)^2} = \frac{1}{3}\sqrt{2005}$.

Also solved by NATALIA DESY, student, Palembang, Indonesia; JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; VEDULA N. MURTY, Dover, PA, USA; and MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB.

5. Evaluate $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$.

Identical solutions by Natalia Desy, student, Palembang, Indonesia; Justin Yang, student, Lord Bing Secondary School, Vancouver, BC; and Vedula N. Murty, Dover, PA, USA.

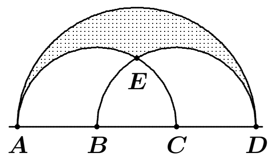
Let $x=\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$. Then

$$x^2 \;=\; 6+\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}} \;=\; 6+x\,,$$

or $x^2-x-6=0$. Factoring, we get $(x-3)(x+2)=0$, which implies that $x=3$ or $x=-2$. But x is the result of a positive square root; thus, $x=3$.

Also solved by MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB.

6. Let A, B, C, and D be collinear points such that $\overline{AB}=\overline{BC}=\overline{CD}=1$. Consider three semi-circles of respective diameters $\overline{AC}$, $\overline{BD}$ and $\overline{AD}$. Let E be the intersection of the semi-circles with centres B and C. Determine the area of the curvilinear triangle AED (shaded in the drawing).



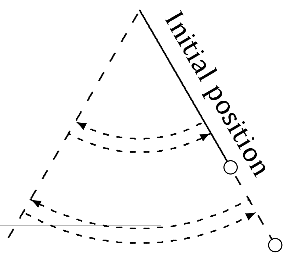
Solution by Justin Yang, student, Lord Bing Secondary School, Vancouver, BC.

Let $[\nabla]$ denote the area of region ∇, and let $\widetilde{XY}$ denote the semi-circle with diameter XY. From the problem statement, we deduce that the large semi-circle has diameter 3 (and radius $\frac{3}{2}$) and that $\triangle BEC$ is equilateral. With this additional information, we can evaluate the required area using the Inclusion-Exclusion Principle:

$$\begin{aligned}
&[\text{curvilinear triangle } AED]\\
&\quad= [\widetilde{AD}]-[\widetilde{AC}]-[\widetilde{BD}]+[\text{curvilinear triangle } BEC]\\
&\quad= \tfrac{1}{2}\left(\tfrac{9}{4}\pi\right)-\tfrac{1}{2}\pi-\tfrac{1}{2}\pi+[\text{curvilinear triangle } BEC]\\
&\quad= \tfrac{1}{8}\pi+[\text{curvilinear triangle } BEC]\\
&\quad= \tfrac{1}{8}\pi+[\text{sector } CBE]+[\text{sector } BCE]-[\triangle BEC]\\
&\quad= \tfrac{1}{8}\pi+\tfrac{1}{6}\pi+\tfrac{1}{6}\pi-\tfrac{1}{4}\sqrt{3} \;=\; \tfrac{11}{24}\pi-\tfrac{1}{4}\sqrt{3}\,.
\end{aligned}$$

Also solved by NATALIA DESY, student, Palembang, Indonesia; JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; and MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB.

7. The oscillation period of a pendulum is proportional to the square root of its length (for example, to triple the oscillation period, we multiply the length by nine). Two pendulums of different lengths are released from the initial position shown. The shorter one measures 25 cm, and its oscillation period is 1 second. The two pendulums are aligned again for the first time after 7 seconds in their initial position. Find the length of the longer pendulum. (Air resistance is neglected.)



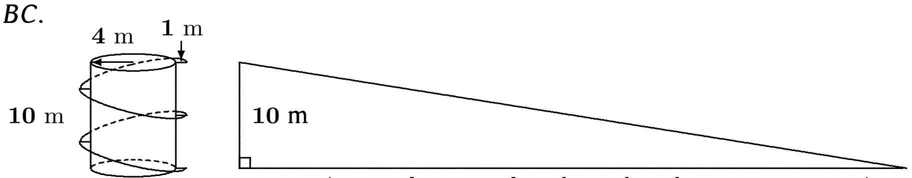
Identical solutions by Natalia Desy, student, Palembang, Indonesia; and Justin Yang, student, Lord Bing Secondary School, Vancouver, BC.

From the shorter pendulum, we have $T = k\sqrt{L}$ or $1 = k\sqrt{25}$, which gives us $k = 1/5$. For the longer pendulum, we get $T = k\sqrt{L}$ or $7 = \frac{1}{5}\sqrt{L}$, which gives us $L = 35^2 = 1225$ cm.

Also solved by MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB. There was one incorrect solution submitted.

8. In a refinery, a cylindrical storage tank has a spiral staircase one meter wide attached to its exterior. The staircase goes from the bottom to the top while making exactly 2 complete revolutions. If the tank has a height of 10 m and a diameter of 8 m, find the length of the exterior edge of the staircase.

Solution by Justin Yang, student, Lord Bing Secondary School, Vancouver, BC.



The length of the exterior edge of the staircase can be thought of as the hypotenuse of a right-angled triangle obtained by "unrolling" the staircase from around the tank. One leg of the triangle is the height of the tank ($H = 10$ m); the other is twice the circumference of a circle with diameter $D = 8 + 2 = 10$ m. (We take twice the circumference to take into account the two complete revolutions the ramp makes around the tank.)

$$\text{Ramp length} = \sqrt{H^2 + (2\pi D)^2} = \sqrt{100 + 400\pi^2} = 10\sqrt{1 + 4\pi^2}\,.$$

Also solved by JONATHAN LOVE, student, Queen Elizabeth Jr.–Sr. High School, Calgary, AB; and MARIYA SARDARLI, student, McKernan Elementary and Junior High School, Edmonton, AB.

That brings us to the end of another issue. This month's winner of a past volume of Mathematical Mayhem is Justin Yang. Congratulations, Justin! Continue sending in your contests and solutions.