Pólya's Paragon

The Pigeonhole Principle

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In problem-solving, we can sometimes get to the answer using the most direct approach (which is often the first one we think of). But there may be approaches to a problem that are *indirect* or *non-constructive*; they force a solution or situation to happen, but not explicitly. In fact, even in cases where a direct attack works, these alternative methods sometimes provide simpler, more elegant solutions. In this issue we will explore one of these ideas and look at a number of problems in which it can be applied.

The simplest version of this idea is easy to explain. Suppose you have 10 balls and 9 boxes, and you must put all the balls into the boxes in some manner. There are of course lots of ways to do this. You could, for instance, put all of the balls in one box and leave the others empty, or you could try to distribute the balls evenly. But, no matter how you do it, *at least one of the boxes must get more than one ball!* This is because there are more balls than boxes.

Now I hope this is clear. Even if you tried to fill the boxes with one ball each, there would still be that one extra ball at the end, and it would need to go somewhere! Once you place it in a box, you must have (at least) two balls in one of the boxes.

It may surprise you that this idea is important enough to have a name. It is called the *pigeonhole principle*. Its name evokes an image of lots of pigeons fighting to get into a smaller number of holes to roost. It simply says that you cannot stuff lots of things into an insufficient number of boxes. A slightly more formal statement might be:

Pigeonhole Principle. If more than n objects (pigeons) are distributed into exactly n boxes (holes), then (at least) one of the boxes must contain more than one of the objects.

If the number of objects is a lot larger than the number of boxes, then we can make slightly stronger conclusions. Suppose we had 19 balls to place in 9 boxes. Can you see why it now must be the case that one (or possibly more) of the boxes must have at least 3 balls? So there is a more general version of the pigeonhole principle:

Pigeonhole Principle (General Version). Let $k \geq 1$. If more than kn objects (pigeons) are distributed into exactly n boxes (holes), then (at least) one of the boxes must contain more than k objects.

Even this generalization seems fairly obvious. What might surprise you is the number of situations in which this principle can be applied. Often there is some subtlety that makes the application not quite immediate. Let's look at some examples.

Example 1: At a conference there are 100 people participating. Show that there must be two of them who know the same number of other participants.

Solution: We will treat the 100 participants as 'pigeons'; we need to put them into 'holes'. But what sort of holes? It seems that we should assign to each participant the number of *other* participants he or she knows. This will be a number between 0 and 99. But wait! That's 100 holes! It seems possible that we might be able to assign all 100 numbers to the 100 different people. The pigeonhole principle does not seem to apply.

There's a subtlety though. Suppose person X receives the number 99. Then this person must know everybody else, and so nobody can be assigned the number 0! But now we must assign each participant a number from 1 to 99, and the pigeonhole principle applies. If no individual gets assigned the number 99, then the 100 people are each assigned one of the 99 numbers 0 through 98, and again we may apply the pigeonhole principle. In any case, two people must have the same number, which means that they know the same number of participants.

Example 2: Let $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$ be the set of prime numbers less than 20. Show that there are four non-empty subsets of S with the same sum.

Solution: We will set up this problem by taking the possible sums to be the 'holes' and the various subsets to be the 'pigeons'. Since S has 8 elements, there are $2^8 - 1 = 255$ non-empty subsets of S. The sums which are possible for non-empty sets lie between 2 (corresponding to the subset $\{2\}$) and 77 (the sum of all the elements of S), for a total of 76 possible values. Since $255 = 3 \cdot 76 + 27$, the general version of the pigeonhole principle applies here with $k = 3$; namely, there must be a sum which corresponds to at least 4 subsets.

Here is an old favourite of mine that completely stumped me once when I was a student. (I wasn't thinking of the pigeonhole principle at the time.)

Example 3: Suppose that each square of a 3×7 chessboard is painted red or black at random. Show that the board must contain a rectangle whose four corner squares are all coloured the same.

Solution: At first glance, this does not look like the sort of problem where the pigeonhole principle would help. The squares look like boxes, but where are the pigeons? You may be tempted, as I was, to start working out the possibilities.

But wait a moment! Let's get a little more creative. Look at the columns of the board. There are 7 columns each containing 3 squares. No matter how the board is painted, each column must contain some pair of squares of the same colour, because there are 3 squares per column and only 2 colours. (We are applying the pigeon-hole principle with the 3 squares as the pigeons and the 2 colours as the holes.)

Now, in order for a rectangle to have its four corners coloured the same colour, there must be two different columns in which squares of the same colour are placed in the same two rows. For example, we might have black squares in rows 1 and 2 in two different columns i and j.

This leads us to consider the possible ways of placing pairs of squares of the same colour in a column of 3 squares. There are $\binom{3}{2} = 3$ ways to place a pair of black squares, and the same number of ways to place a pair of red squares. Thus, there are $2\binom{3}{2} = 6$ ways altogether. Since there are 7 columns in the board, there must be (at least) 2 different columns in which a pair of squares of the same colour are placed in the same way. Here we are applying the pigeonhole principle with the columns as the pigeons and the possible ways of placing a pair of like-coloured squares in a column as the holes. Now that's subtle! But it leads to the desired conclusion: the board must contain a rectangle whose four corner squares are painted with the same colour.

Problems for further study:

I now offer you a few problems to try out. Remember to keep in mind the idea of distributing things. Be on the lookout for the 'pigeons' you're trying to distribute and the 'holes' into which they are going. Identifying these may require a little creativity on your part. Good luck! Feel free to contact me for further discussion of your solutions (`jeff.hooper@acadiau.ca`).

1. Suppose we distribute 5 points in the interior of a square S of side length 2. Prove that some pair of these points must have distance less than $\sqrt{2}$.

2. Take any set A consisting of 10 natural numbers between 1 and 99. Show that there must be two disjoint subsets of the set A which have the same sum.

3. Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Show that there must be two distinct integers in A which sum to 104.

4. Suppose that 5 points are placed randomly on a sphere. Show that there must be a hemisphere which contains at least 4 of them.

5. Let x be any real number, and let $A = \{x, 2x, 3x, 4x, \ldots, (n-1)x\}$. Show that there must be at least one number in the set A which differs from an integer by at most $1/n$.

6. Suppose that k colours are available to paint the squares of a $(k+1) \times n$ chessboard. What is the largest value of n, in terms of k, for which the board can be painted in such a way that there is no rectangle whose four corner squares have the same colour?