MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

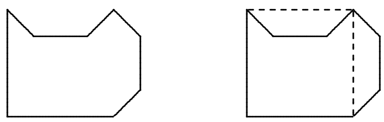
Please send your solutions to the problems in this edition by **1 August 2007**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M288. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The following figure can be cut into two pieces and reassembled into a square, by simply cutting off the 'tab' and placing it in the cutaway at the top, as shown in the second image.



Determine a method to cut the given figure into three pieces which can be reassembled to form a square. (Find a method which is essentially different from cutting it into two pieces; for example, cutting the tab into two pieces would not be considered different from the two-piece dissection.)

M289. *Proposed by K.R.S. Sastry, Bangalore, India.*

Solve the following equation for real x:

$$\log\left(x+\sqrt{5x-\tfrac{13}{4}}\right) \;=\; -\log\left(x-\sqrt{5x-\tfrac{13}{4}}\right)\,.$$

M290. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Give a purely geometric proof that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$.

M291. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

The right triangle having sides 3, $\sqrt{7}$, and 4, has the strange property that the two integer lengths sum to the value under the square root sign for the length of the third side.

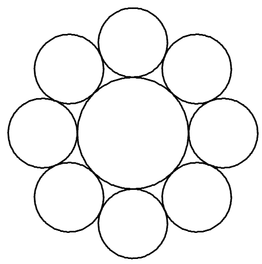
1. Find another such triangle.
2. Prove that there are infinitely many such triangles, and show how to construct them.
3. Does the formula work only for integers?

M292. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let x be a positive number. Prove that $\sqrt{\dfrac{[x]}{x+\{x\}}} + \sqrt{\dfrac{\{x\}}{x+[x]}} > 1$, where $[x]$ and $\{x\}$ represent the integer part and the fractional part of x, respectively.

M293. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

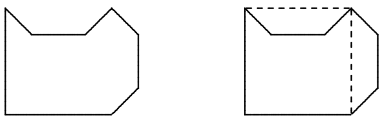
Eight equal circles are mutually tangent in pairs and tangent externally to a unit circle. Determine the common radii of the eight smaller circles.



. . . .

M288. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

La figure ci-dessous peut être coupée en deux morceaux qu'on peut réarranger pour former un carré, comme le montre le second dessin.



Trouver une méthode pour couper la figure donnée en trois morceaux pouvant former un carré par réarrangement. (Cette méthode doit être essentiellement différente de la première ; simplement couper en deux le morceau ajouté pour former le premier carré ne compte pas.)

M289. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Trouver les solutions réelles de l'équation :

$$\log\left(x+\sqrt{5x-\tfrac{13}{4}}\right) \;=\; -\log\left(x-\sqrt{5x-\tfrac{13}{4}}\right)\;.$$

M290. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Trouver une démonstration purement géométrique de l'égalité $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$.

M291. *Proposé par Robert Bilinski, Collège Montmorency, Laval, QC.*

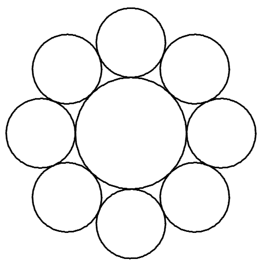
Le triangle rectangle de côtés 3, $\sqrt{7}$ et 4 possède la curieuse propriété qu'un de ses côtés est la racine carrée de la somme des côtés mesurés par des entiers.

1. Trouver un autre tel triangle.
2. Montrer qu'il existe une infinité de tels triangles et décrire leur construction.
3. La formule n'est-elle valable que pour des entiers ?

M292. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit x un nombre positif. Montrer que $\sqrt{\dfrac{[x]}{x+\{x\}}}+\sqrt{\dfrac{\{x\}}{x+[x]}}>1$, où $[x]$ et $\{x\}$ désignent respectivement les parties entière et fractionnaire de x.

M293. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*



On couronne le cercle unité avec huit petits cercles égaux tangents et tangents deux à deux. Trouver leur rayon commun.

Mayhem Solutions

M238. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Soit PQ une corde d'une parabole et soit R le point milieu de PQ. Soit S un point sur la parabole tel que la tangente en S est parallèle à PQ. Si T désigne le point d'intersection des tangentes en P et Q, montrer que R, S et T sont colinéaires.

Solution par Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

Plaçons la parabole et les points sur un plan cartésien. On choisit nos axes tels que le sommet de la parabole se situe à l'origine. Soit la parabole d'équation $y = ax^2$ et les points $P(p, ap^2)$ et $Q(q, aq^2)$. On va prouver que, sous ces conditions, les points R, S, et T ont la même abscisse, et sont par le fait même colinéaires.

Soit r l'abscisse du point R. Puisque R est le point milieu de la corde joignant P et Q, son abscisse est donc la moyenne de celles des points P et Q ; c'est-à-dire $r = \frac{1}{2}(p+q)$.

Soit s l'abscisse du point S. Puisque la tangente en S et la corde PQ sont parallèles, elles ont la même pente. Nous avons donc l'équation suivante à résoudre pour s :

$$\left.\frac{d(ax^2)}{dx}\right|_{x=s} = 2as = \frac{aq^2 - ap^2}{q-p} = \frac{a(q^2-p^2)}{q-p}\,.$$

Puisque $p \neq q$, on a $2as = a(p+q)$ et on obtient $s = \frac{1}{2}(p+q)$.

Maintenant, on va trouver l'équation des tangentes aux points P et Q. Les dérivées de la parabole aux points P et Q nous donneront les pentes de ces tangentes. Ainsi, les équations des tangentes sont :

$$y = 2apx - ap^2 \quad \text{et} \quad y = 2aqx - aq^2\,.$$

Pour trouver l'abscisse t du point T, on résout l'équation suivante pour t :

$$\begin{aligned} 2apt - ap^2 &= 2aqt - aq^2\,, \\ 2at(p-q) &= a(p^2-q^2)\,. \end{aligned}$$

Ainsi, $t = \frac{1}{2}(p+q)$, parce que $p \neq q$.

Donc, les points R, S et T ont la même abscisse $\frac{1}{2}(p+q)$ et sont colinéaires.

En outre résolu par HASAN DENKER, Istanbul, Turquie ; et TITU ZVONARU, Comăneşti, Roumanie. Une solution incorrecte a aussi été soumise.

M239. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

If a, b, $c > 0$, prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{(a+b+c)^2}{6abc}\,.$$

Solution by Vedula N. Murty, Dover, PA, USA.

We have $(a+b)^2 - 4ab = (a-b)^2 \geq 0$, and hence, $4ab \leq (a+b)^2$. Similarly, $4bc \leq (b+c)^2$ and $4ca \leq (c+a)^2$. Therefore,

$$\begin{aligned} 4abc\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) &= \frac{4ab}{a+b}\,c + \frac{4bc}{b+c}\,a + \frac{4ca}{c+a}\,b \\ &\leq (a+b)c + (b+c)a + (c+a)b \\ &= 2(ab+bc+ca)\,. \qquad (1) \end{aligned}$$

Using the well-known inequality $ab + bc + ca \leq a^2 + b^2 + c^2$, we obtain

$$\begin{aligned} 3(ab+bc+ca) &\leq a^2 + b^2 + c^2 + 2(ab+bc+ca) \\ &= (a+b+c)^2\,. \qquad (2) \end{aligned}$$

Combining (1) and (2), we have

$$4abc\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \leq \frac{2}{3}(a+b+c)^2\,.$$

Dividing by $4abc$ gives the desired result.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; MIHÁLY BENCZE, Brasov, Romania; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, Cégep de Drummondville, Drummondville, QC; BABIS STERGIOU, Chalkida, Greece; and TITU ZVONARU, Comăneşti, Romania.

M240. *Proposé par l'Équipe de Mayhem.*

En utilisant une seule fois chacun des chiffres de 0 à 9, trouver quatre carrés parfaits (positifs) tels qu'il y en ait un de quatre chiffres, un de trois, un de deux et un dernier de un chiffre. (Note : Il y a plus d'une solution. Combien pouvez-vous en trouver ?)

Solution par Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

Évidemment, nous ne devons pas considérer les carrés qui comportent au moins 2 nombres identiques. Sous cette condition, on peut démontrer qu'aucun carré de 1, 2, ou 3 chiffres ne contienne de 0. En rédigeant une table comprenant tous les nombres carrés à considérer (de 1, 2, 3, ou 4 chiffres, sans répétition, et comprenant le chiffre 0 dans le cas des nombres à 4 chiffres), on obtient 37 nombres.

Pour chaque carré à 4 chiffres, on inscrit les carrés à 2 chiffres possibles (ceux dont les chiffres n'apparaissent pas dans le carré à 4 chiffres). Pour chaque paire, on inscrit ensuite les carrés à 1 chiffre qui n'apparaissent pas dans le carré à 4 chiffres ou dans celui à 2 chiffres. En vérifiant si il est possible de trouver un carré à 3 chiffres comprenant les 3 chiffres non-utilisés, on obtient les 4 solutions suivantes : $(9, 81, 576, 2304)$, $(9, 16, 784, 3025)$, $(9, 81, 324, 7056)$, et $(1, 36, 784, 9025)$.

Autres solutions soumises par HASAN DENKER, Istanbul, Turquie ; et TITU ZVONARU, Comăneşti, Roumanie. Une solution incomplète a aussi été soumise.

M241. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Three gunfighters, called Quick, Fast, and Slow, stand one at each vertex of an equilateral triangle. Quick is faster on the draw than Fast, and Fast is faster than Slow. If x intends to fire at y, we will say that x *targets* y. We will assume that if x fires at y, then y will be hit, and that if x and y both target each other, the one who is slower on the draw will be hit before he can fire. A combatant cannot fire once he has been hit.

In the first phase of the confrontation, each combatant targets one of the other two and fires a maximum of one round. No man knows how fast the other two are, and the targeting choices are made randomly and cannot be changed during the first phase.

If two combatants survive the first phase, they face each other in a second phase and the fastest draw wins. If only one combatant survives the first phase, he is the winner (and there is no second phase).

Find the probability that:

(a) Quick survives; (b) Fast survives; (c) Slow survives.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

There are 8 targeting possibilities in the first round as shown by the table of outcomes below, where Quick, Fast, and Slow are denoted by Q, F, and S, respectively.

Targets for Q, F, and S	[*Ed.:* First Round Survivor]	Final Survivor
F, Q, Q	Slow	Slow
F, Q, F	Quick and Slow	Quick
F, S, Q	None	None
F, S, F	Quick	Quick
S, Q, Q	Fast	Fast
S, Q, F	None	None
S, S, Q	Quick and Fast	Quick
S, S, F	Quick and Fast	Quick

Thus, the probability of survival for Quick is $\frac{1}{2}$, for Fast is $\frac{1}{8}$, and for Slow is $\frac{1}{8}$.

Also solved by Jean-David Houle, Cégep de Drummondville, Drummondville, QC. A solution submitted by Hasan Denker, Istanbul, Turkey used the assumption that the one who is slower on the draw will always be hit before he can fire. In that case, the probability that Quick survives is $\frac{1}{2}$, the probability that Fast survives is $\frac{1}{4}$, and the probability that Slow survives is $\frac{1}{4}$.

M242. *Proposé par Houda Anoun, Bordeaux, France.*

Pour quels nombres naturels x le nombre $x^4 + x^3 + x^2 + x + 1$ est-il un carré parfait ?

Solution par Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

Disons que $f(x) = x^4 + x^3 + x^2 + x + 1$.

Cas 1. x est un nombre pair.

Notons que

$$\begin{aligned} \left(x^2 + \tfrac{1}{2}x\right)^2 &= x^4 + x^3 + \tfrac{1}{4}x^2 \;<\; f(x) \\ \text{et} \quad \left(x^2 + \tfrac{1}{2}x + 1\right)^2 &= x^4 + x^3 + \tfrac{9}{4}x^2 + x + 1 \;\geq\; f(x)\,, \end{aligned}$$

alors $\left(x^2 + \frac{1}{2}x\right)^2 < f(x) \leq \left(x^2 + \frac{1}{2}x^2 + 1\right)^2$.

L'égalité survient si $x = 0$. Dans tous les autres cas, le polynôme $f(x)$ est compris entre deux carrés parfaits consécutifs et ne peut donc pas être, lui aussi, un carré parfait.

Cas 2. x est un nombre impair.

Pour $x \geq 5$, on a

$$\begin{aligned} \left(x^2 + \tfrac{1}{2}x - \tfrac{1}{2}\right)^2 &= x^4 + x^3 - \tfrac{3}{4}x^2 - \tfrac{1}{2}x + \tfrac{1}{4} \;<\; f(x) \\ \text{et} \quad \left(x^2 + \tfrac{1}{2}x + \tfrac{1}{2}\right)^2 &= x^4 + x^3 + \tfrac{5}{4}x^2 + \tfrac{1}{2}x + \tfrac{1}{4} \;>\; f(x)\,, \end{aligned}$$

donc $\left(x^2 + \frac{1}{2}x - \frac{1}{2}\right)^2 < f(x) < \left(x^2 + \frac{1}{2}x + \frac{1}{2}\right)^2$.

Le polynôme $f(x)$ est compris entre deux carrés parfaits consécutifs et ne peut donc pas être, lui aussi, un carré parfait.

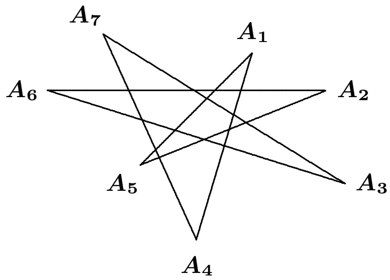
Ceci étant dit, il ne reste qu'à vérifier les valeurs de x qui sont impaires et inférieures à 5. Si $x = 1$, alors $x^4 + x^3 + x^2 + x + 1 = 5$, qui n'est pas un carré parfait. Si $x = 3$, alors $x^4 + x^3 + x^2 + x + 1 = 121 = 11^2$, qui donne une solution.

Le seul nombre naturel x satisfaisant l'énoncé est $x = 3$.

Autres solutions soumises par ALINA ALT et ARKADY ALT, San José, CA, É-U; RICHARD I. HESS, Rancho Palos Verdes, CA, É-U; EDWARD T.H. WANG, Université Wilfrid Laurier, Waterloo, ON; et TITU ZVONARU, Comăneşti, Roumanie. Une solution incomplète a aussi été soumise.

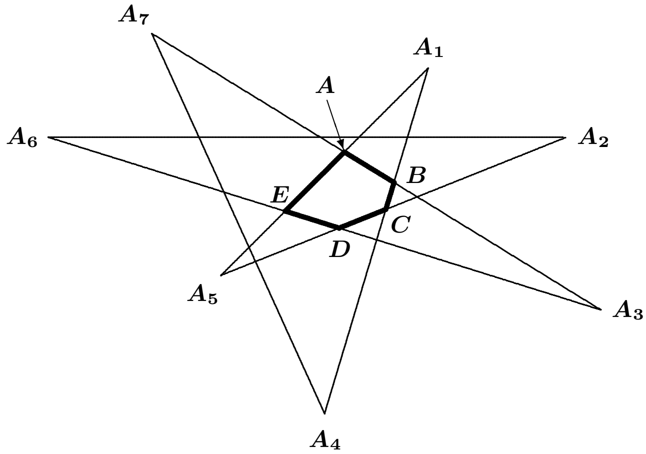
M243. *Proposed by K.R.S. Sastry, Bangalore, India.*

In the 7-point star shown, no three lines are concurrent. Find the sum $A_1 + A_2 + \ldots + A_7$.



Solution by Hasan Denker, Istanbul, Turkey.

The given 7-point star, with no three lines concurrent, generates pentagon $ABCDE$, as shown.



Considering triangles AEA_3, A_7A_4B, A_1A_5C, and A_2A_6D, the following relationships are obtained:

$$\begin{aligned} A + E &= 180^\circ - A_3\,, \\ B &= 180^\circ - A_4 - A_7\,, \\ C &= 180^\circ - A_1 - A_5\,, \\ D &= 180^\circ - A_2 - A_6\,. \end{aligned}$$

Summing these equations, we can then conclude that

$$\begin{aligned} &A + B + C + D + E \\ &\qquad = (180^\circ - A_3) + (180^\circ - A_4 - A7) \\ &\qquad\qquad + (180^\circ - A_1 - A_5) + (180^\circ - A_2 - A_6) \\ &\qquad = 720^\circ - (A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7)\,. \end{aligned}$$

However, $A + B + C + D + E = 540^\circ$. Hence,

$$540^\circ = 720^\circ - (A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7)\,.$$

Therefore, $A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 = 180^\circ$.

There was one incomplete solution submitted.

Problem of the Month

Ian VanderBurgh

This month, we give some thought about repeating decimals (decimals, decimals, decimals, decimals...)

Problem #1 (2006 Canadian Open Mathematics Challenge)
Suppose n and D are integers with n positive and $0 \le D \le 9$. Determine n if $\frac{n}{810} = 0.\overline{9D5} = 0.9D59D59D5\ldots$.

I knew that I should have paid more attention in elementary school! If you're like me, you probably remember that $\frac{1}{3} = 0.33333\ldots$ which can also be written as $0.\overline{3}$. Maybe you remember that $\frac{7}{9} = 0.\overline{7} = 0.7777\ldots$. How about $\frac{1}{7}$? Do you know that this equals $0.\overline{142857}$?

To solve Problem #1, it would be helpful to convert the repeating decimal to a fraction. But how do we do this? Let's look at two different ways.

In the first approach, we set $x = 0.\overline{9D5} = 0.9D59D59D5\ldots$. Then $1000x = 9D5.9D59D5\ldots = 9D5.\overline{9D5}$. Thus,

$$\begin{aligned} 999x &= 1000x - x = 9D5.\overline{9D5} - 0.\overline{9D5} = 9D5 \qquad \text{(an integer!)} \\ x &= \frac{9D5}{999}. \end{aligned}$$

Does this method look familiar? It may, if you have ever tried to prove that $0.\overline{9}$ actually equals 1.

For a second approach, we rewrite $0.9D59D59D5\ldots$ as

$$\frac{9D5}{10^3} + \frac{9D5}{10^6} + \frac{9D5}{10^9} + \cdots$$

This is an infinite geometric series with first term $a = \frac{9D5}{10^3}$ and common ratio $r = \frac{1}{10^3}$; thus, its sum is

$$\frac{a}{1-r} = \frac{\frac{9D5}{10^3}}{1 - \frac{1}{10^3}} = \frac{9D5}{1000-1} = \frac{9D5}{999}.$$

It is reassuring to get the same answer in two different ways. Try using one or both of these methods to show that $0.\overline{1234} = \frac{1234}{9999}$ and $0.\overline{abc} = \frac{abc}{999}$. Can you come up with some general rules for converting repeating decimals to fractions?

Now we are ready to solve Problem #1.

Solution to Problem #1: From the given information and our comments above, we know that $\frac{n}{810} = 0.\overline{9D5} = \frac{9D5}{999}$. Clearing the fractions yields

$999n = 810(9D5)$. We can simplify this by dividing both sides by 27, giving $37n = 30(9D5)$. Since 37 is a factor of the left side, it must be a factor of the right side. Since 37 and 30 have no common factors, then 37 must divide exactly into $9D5$. How can we determine D? One way would be to get out a calculator and try to find a multiple of 37 that is between 900 and 1000 and ends with a 5. This wouldn't be too hard.

Here is another approach. We first note that

$$37 \times 20 = 740 < 9D5 < 1120 = 37 \times 30 .$$

Hence, $9D5 = 37 \times 25$ because no other number in the permissible range when multiplied by 37 will end in 5. Therefore, $D = 2$.

But we want the value of n. Recall that $37n = 30(9D5) = 30(925)$. Hence, $n = 30(925)/37 = 750$.

As with many problems involving a repeating decimal, the decimal gets converted to a fraction. So the amount of knowledge of repeating decimals that we need is not enormous.

Here is another such problem to keep you busy over the next month:

Problem #2 (1992 AIME) Let S be the set of all rational numbers r with $0 < r < 1$, that have a repeating decimal expansion of the form $0.\overline{abc}$, where the digits a, b, and c are not necessarily distinct. To write the elements of S as fractions in lowest terms, how many different numerators are required?

Good luck! I've put a few hints at the end.

In February's Problem of the Month, we looked at a problem involving determining the average number of "change points" in sequences of 0s and 1s. This involved counting the total number of change points over all such sequences in a clever way.

Imagine my surprise on the first Saturday in December (about the time I was writing the February column), when I saw the following problem on the 2006 William Lowell Putnam Mathematical Competition. (I have modified this problem slightly to remove the special case of $n = 2$ and to remove some of the more technical notation.)

Problem #3. A permutation π of $\{1, 2, \ldots, n\}$ (with $n \geq 3$) has a local maximum at position k if the two neighbouring numbers (or, in case $k = 1$ or $k = n$, the one neighbouring number) are both smaller than the number in position k. (For example, if $n = 5$, then $2, 1, 4, 5, 3$ has local maxima in positions 1 and 4.) What is the average number of local maxima of a permutation of $\{1, 2, \ldots, n\}$, averaging over all such permutations?

We will try to solve this problem by the same technique that we used in February: fixing a position and counting the total number of permutations with a local maximum in that position.

Solution to Problem #3: First consider position 1. How many permutations have a local maximum in position 1? Whether or not there is a local maximum at position 1 depends on the numbers in positions 1 and 2. Any pair of numbers can give a local maximum at position 1 if they are arranged with the larger number first. (For example, the pair 3 and 5 gives a local maximum in position 1 if the 5 comes before the 3.)

There are $\binom{n}{2}$ possible pairs of numbers that can be placed in positions 1 and 2. There is only one way to arrange a given pair to get a local maximum in position 1. There are then $(n-2)!$ ways of filling out the rest of the permutation. Thus, there are

$$\binom{n}{2}(n-2)! \;=\; \frac{n!}{(n-2)!2!}(n-2)! \;=\; \frac{n!}{2}$$

permutations with a local maximum in position 1. In other words, among all such permutations, there are $\frac{1}{2}n!$ local maxima in position 1. By a similar argument, there are $\frac{1}{2}n!$ local maxima in position n.

Now consider a position k with $1 < k < n$. How many local maxima are there at position k? Whether there is a local maximum at position k depends on the numbers in positions $k-1$, k, and $k+1$. Any triple of numbers can be arranged to form a local maximum at position k in two ways. For example, if we choose 1, 3, 7, then a local maximum occurs in the middle if (and only if) they are arranged as 1, 7, 3 or 3, 7, 1. There are $\binom{n}{3}$ ways of choosing the three numbers that will go in positions $k-1$ through $k+1$, two ways of arranging these numbers to form a local maximum at position k, and $(n-3)!$ ways to arrange the remaining $n-3$ numbers in the permutation. Thus, there are

$$2\binom{n}{3}(n-3)! \;=\; \frac{2n!}{(n-3)!3!}(n-3)! \;=\; \frac{n!}{3}$$

permutations with a local maximum at position k. In other words, there are $\frac{1}{3}n!$ local maxima at position k among all such permutations. (Remember that there are $n-2$ values for k that we have to keep track of in this case.)

Hence, the total number of local maxima over all such permutations is

$$\tfrac{1}{2}n! + \tfrac{1}{2}n! + (n-2)\left(\tfrac{1}{3}n!\right) \;=\; \tfrac{1}{3}(n+1)n!\,.$$

Since the total number of permutations of $\{1, 2, \ldots, n\}$ is $n!$, the average number of local maxima is $\frac{1}{3}(n+1)$.

It's always neat to see an old technique come in handy. That's part of the reason why we practice solving problems—the more we practice, the more techniques we learn, and the more likely we are to think, "Hey, wait a second! I know what to do here."

Hints for Problem #2:

- Convert the repeating decimal to a fraction.
- When is this fraction irreducible? How many of these cases are there?
- If the given fraction is reducible, what happens? What are the possible denominators when reduced? What are the possible numerators?

Pólya's Paragon

The Pigeonhole Principle

Jeff Hooper

In problem-solving, we can sometimes get to the answer using the most direct approach (which is often the first one we think of). But there may be approaches to a problem that are *indirect* or *non-constructive*; they force a solution or situation to happen, but not explicitly. In fact, even in cases where a direct attack works, these alternative methods sometimes provide simpler, more elegant solutions. In this issue we will explore one of these ideas and look at a number of problems in which it can be applied.

The simplest version of this idea is easy to explain. Suppose you have 10 balls and 9 boxes, and you must put all the balls into the boxes in some manner. There are of course lots of ways to do this. You could, for instance, put all of the balls in one box and leave the others empty, or you could try to distribute the balls evenly. But, no matter how you do it, *at least one of the boxes must get more than one ball!* This is because there are more balls than boxes.

Now I hope this is clear. Even if you tried to fill the boxes with one ball each, there would still be that one extra ball at the end, and it would need to go somewhere! Once you place it in a box, you must have (at least) two balls in one of the boxes.

It may surprise you that this idea is important enough to have a name. It is called the *pigeonhole principle*. Its name evokes an image of lots of pigeons fighting to get into a smaller number of holes to roost. It simply says that you cannot stuff lots of things into an insufficient number of boxes. A slightly more formal statement might be:

Pigeonhole Principle. If more than n objects (pigeons) are distributed into exactly n boxes (holes), then (at least) one of the boxes must contain more than one of the objects.

If the number of objects is a lot larger than the number of boxes, then we can make slightly stronger conclusions. Suppose we had 19 balls to place in 9 boxes. Can you see why it now must be the case that one (or possibly more) of the boxes must have at least 3 balls? So there is a more general version of the pigeonhole principle:

Pigeonhole Principle (General Version). Let $k \geq 1$. If more than kn objects (pigeons) are distributed into exactly n boxes (holes), then (at least) one of the boxes must contain more than k objects.

Even this generalization seems fairly obvious. What might surprise you is the number of situations in which this principle can be applied. Often there is some subtlety that makes the application not quite immediate. Let's look at some examples.

Example 1: At a conference there are 100 people participating. Show that there must be two of them who know the same number of other participants.

Solution: We will treat the 100 participants as 'pigeons'; we need to put them into 'holes'. But what sort of holes? It seems that we should assign to each participant the number of *other* participants he or she knows. This will be a number between 0 and 99. But wait! That's 100 holes! It seems possible that we might be able to assign all 100 numbers to the 100 different people. The pigeonhole principle does not seem to apply.

There's a subtlety though. Suppose person X receives the number 99. Then this person must know everybody else, and so nobody can be assigned the number 0! But now we must assign each participant a number from 1 to 99, and the pigeonhole principle applies. If no individual gets assigned the number 99, then the 100 people are each assigned one of the 99 numbers 0 through 98, and again we may apply the pigeonhole principle. In any case, two people must have the same number, which means that they know the same number of participants.

Example 2: Let $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$ be the set of prime numbers less than 20. Show that there are four non-empty subsets of S with the same sum.

Solution: We will set up this problem by taking the possible sums to be the 'holes' and the various subsets to be the 'pigeons'. Since S has 8 elements, there are $2^8 - 1 = 255$ non-empty subsets of S. The sums which are possible for non-empty sets lie between 2 (corresponding to the subset $\{2\}$) and 77 (the sum of all the elements of S), for a total of 76 possible values. Since $255 = 3 \cdot 76 + 27$, the general version of the pigeonhole principle applies here with $k = 3$; namely, there must be a sum which corresponds to at least 4 subsets.

Here is an old favourite of mine that completely stumped me once when I was a student. (I wasn't thinking of the pigeonhole principle at the time.)

Example 3: Suppose that each square of a 3×7 chessboard is painted red or black at random. Show that the board must contain a rectangle whose four corner squares are all coloured the same.

Solution: At first glance, this does not look like the sort of problem where the pigeonhole principle would help. The squares look like boxes, but where are the pigeons? You may be tempted, as I was, to start working out the possibilities.

But wait a moment! Let's get a little more creative. Look at the columns of the board. There are 7 columns each containing 3 squares. No matter how the board is painted, each column must contain some pair of squares of the same colour, because there are 3 squares per column and only 2 colours. (We are applying the pigeon-hole principle with the 3 squares as the pigeons and the 2 colours as the holes.)

Now, in order for a rectangle to have its four corners coloured the same colour, there must be two different columns in which squares of the same colour are placed in the same two rows. For example, we might have black squares in rows 1 and 2 in two different columns i and j.

This leads us to consider the possible ways of placing pairs of squares of the same colour in a column of 3 squares. There are $\binom{3}{2} = 3$ ways to place a pair of black squares, and the same number of ways to place a pair of red squares. Thus, there are $2\binom{3}{2} = 6$ ways altogether. Since there are 7 columns in the board, there must be (at least) 2 different columns in which a pair of squares of the same colour are placed in the same way. Here we are applying the pigeonhole principle with the columns as the pigeons and the possible ways of placing a pair of like-coloured squares in a column as the holes. Now that's subtle! But it leads to the desired conclusion: the board must contain a rectangle whose four corner squares are painted with the same colour.

Problems for further study:

I now offer you a few problems to try out. Remember to keep in mind the idea of distributing things. Be on the lookout for the 'pigeons' you're trying to distribute and the 'holes' into which they are going. Identifying these may require a little creativity on your part. Good luck! Feel free to contact me for further discussion of your solutions (`jeff.hooper@acadiau.ca`).

1. Suppose we distribute 5 points in the interior of a square S of side length 2. Prove that some pair of these points must have distance less than $\sqrt{2}$.

2. Take any set A consisting of 10 natural numbers between 1 and 99. Show that there must be two disjoint subsets of the set A which have the same sum.

3. Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Show that there must be two distinct integers in A which sum to 104.

4. Suppose that 5 points are placed randomly on a sphere. Show that there must be a hemisphere which contains at least 4 of them.

5. Let x be any real number, and let $A = \{x, 2x, 3x, 4x, \ldots, (n-1)x\}$. Show that there must be at least one number in the set A which differs from an integer by at most $1/n$.

6. Suppose that k colours are available to paint the squares of a $(k+1) \times n$ chessboard. What is the largest value of n, in terms of k, for which the board can be painted in such a way that there is no rectangle whose four corner squares have the same colour?