SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of Peter Y. Woo, Biola University, La Mirada, CA, USA from the list of solvers of 3102.

3125. [2006 : 111] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let m_a, h_a, and w_a denote the lengths of the median, the altitude, and the internal angle bisector, repectively, to side a in $\triangle ABC$. Define m_b, m_c, h_b, h_c, w_b, and w_c similarly. Let R be circumradius of $\triangle ABC$.

(a) Show that
$$\sum_{\text{cyclic}} \frac{b^2+c^2}{m_a} \le 12R\,.$$

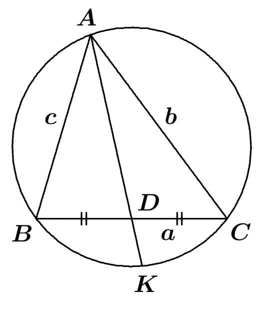
(b) Show that
$$\sum_{\text{cyclic}} \frac{b^2+c^2}{h_a} \ge 12R\,.$$

(c)⋆ Determine the range of
$$\frac{1}{R}\sum_{\text{cyclic}} \frac{b^2+c^2}{w_a}\,.$$

Solution by C.R. Pranesachar, Indian Institute of Science, Bangalore, India, modified by the editor.

(a) Let the median through A meet BC at its mid-point D and meet the circumcircle of triangle ABC at the point K, as shown in the diagram. By Apollonius' Theorem,

$$\begin{aligned} b^2+c^2 &= 2(AD^2+BD^2) \\ &= 2(AD^2+BD\cdot DC) \\ &= 2(AD^2+AD\cdot DK) \\ &= 2AD\cdot(AD+DK) \\ &= 2m_a\cdot AK \\ &\le 2m_a\cdot 2R \;=\; 4Rm_a\,. \end{aligned}$$



Thus, $\frac{b^2+c^2}{m_a} \le 4R$; whence, $\sum_{\text{cyclic}} \frac{b^2+c^2}{m_a} \le 3(4R) = 12R$.

(b) Since $h_a = \dfrac{2[ABC]}{a}$ and $R = \dfrac{abc}{4[ABC]}$, we obtain

$$\frac{b^2+c^2}{h_a} \;=\; 4R\left(\frac{b^2+c^2}{2bc}\right) \;\geq\; 4R\,,$$

which implies that $\sum_{\text{cyclic}} \dfrac{b^2+c^2}{h_a} \geq 3(4R) = 12R$.

(c) Let $s_w = \dfrac{1}{R}\sum_{\text{cyclic}} \dfrac{b^2+c^2}{w_a}$. We will prove that the range of s_w is $(4,\infty)$.

Using the formula $w_a = \dfrac{2bc}{b+c}\cdot\cos\dfrac{A}{2}$, we obtain

$$\frac{b^2+c^2}{Rw_a} \;=\; \frac{b+c}{R\cos\frac{A}{2}}\left(\frac{b^2+c^2}{2bc}\right) \;\geq\; \frac{b+c}{R\cos\frac{A}{2}}\,.$$

Since $b = 2R\sin B$ and $c = 2R\sin C$ (by the extended Law of Sines), we get

$$\frac{b^2+c^2}{Rw_a} \;\geq\; \frac{2(\sin B+\sin C)}{\cos\frac{A}{2}} \;=\; \frac{4\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{A}{2}} \;=\; 4\cos\frac{B-C}{2}\,.$$

Hence,

$$s_w \;\geq\; 4\sum_{\text{cyclic}}\cos\frac{B-C}{2}\,.$$

Since $\cos x \geq 1 - 2x/\pi$ for $0 \leq x \leq \pi/2$, we have

$$\sum_{\text{cyclic}}\cos\frac{B-C}{2} \;=\; \sum_{\text{cyclic}}\cos\frac{|B-C|}{2} \;\geq\; \sum_{\text{cyclic}}\left(1-\frac{1}{\pi}|B-C|\right)\,.$$

Without loss of generality, assume that $C \leq B \leq A$. Then

$$\begin{aligned}\sum_{\text{cyclic}}\cos\frac{B-C}{2} \;&\geq\; 3-\frac{1}{\pi}(B-C-C+A+A-B)\\ &=\; 3-\frac{2}{\pi}(A-C) \;>\; 3-\frac{2}{\pi}\pi \;=\; 1\,.\end{aligned}$$

Thus $s_w > 4$.

Taking $a = 2$ and $b = c = x+1$, where $x \in (0,\infty)$, we obtain

$$s_w \;=\; 4+\frac{(x^2+2x+5)(x+3)\sqrt{2x}}{(x+1)^{\frac{5}{2}}}\,.$$

This is a continuous function of x on $(0,\infty)$ and has the limits 4 and ∞ as x tends to 0 and ∞, respectively. Therefore, the range of s_w is $(4,\infty)$.

Also solved by ARKADY ALT, San Jose, CA, USA (parts (a) *and* (b)*); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (parts* (a) *and* (b)*); MICHEL BATAILLE, Rouen, France (parts* (a) *and* (b)*); FRANCISCO BELLOT ROSADO,* *I.B. Emilio Ferrari, Valladolid, Spain* (*parts* (a) *and* (b))*; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain* (*parts* (a), (b), *and* (c))*; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia* (*part* (b) *only*)*; VEDULA N. MURTY, Dover, PA, USA* (*parts* (a) *and* (b))*; JOEL SCHLOSBERG, Bayside, NY, USA* (*part* (b) *only*)*; PETER Y. WOO, Biola University, La Mirada, CA, USA* (*parts* (a) *and* (b))*; LI ZHOU, Polk Community College, Winter Haven, FL, USA* (*parts* (a) *and* (b))*; and the proposer* (*parts* (a) *and* (b))*. There were also three incomplete solutions to part* (c) *of the problem.*

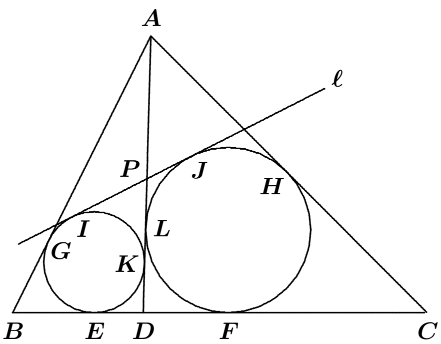
Pranesachar's solution contained some additional detail which has not been included in the modified version above. He proved that the range of the sum $\frac{1}{R}\sum\limits_{\text{cyclic}}\frac{b^2+c^2}{h_a}$ *in part* (a) *is* $(0,12]$ *and the range of* $\frac{1}{R}\sum\limits_{\text{cyclic}}\frac{b^2+c^2}{w_a}$ *in part* (b) *is* $[12,\infty)$.

3126. [2006 : 171, 174; corrected 2006 : 303, 306] *Proposed by Hidetoshi Fukugawa, Kani, Gifu, Japan.*

Let D be any point on the side BC of triangle ABC. Let Γ_1 and Γ_2 be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Let ℓ be the common external tangent to Γ_1 and Γ_2 which is different from BC. If P is the point of intersection of AD and ℓ, show that $2AP = AB + AC - BC$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Suppose that Γ_1 is tangent to BC, AB, ℓ, and AD, at E, G, I, and K, respectively, and that Γ_2 is tangent to BC, AC, ℓ, and AD, at F, H, J, and L, respectively. Then,



$$\begin{aligned}
&AB + AC - BC \\
&\quad = AG + AH - EF \\
&\quad = AK + AL - IJ \\
&\quad = 2AP + PK + PL - PI - PJ \\
&\quad = 2AP\,.
\end{aligned}$$

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; JOHN G. HEUVER, Grande Prairie, AB; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, student, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer.

This problem generated many different solutions, but Zhou's was the neatest. Konečný noted that a very similar problem was in the 20th USA Mathematical Olympiad. Three readers did not see the printed correction to the problem and simply pointed out that there was something wrong.

3127. [2004–075] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let H be the foot of the altitude from A to BC, where BC is the longest side of $\triangle ABC$. Let R, R_1, and R_2 be the circumradii of $\triangle ABC$, $\triangle ABH$, and $\triangle ACH$, respectively. Similarly, let r, r_1, r_2 be the inradii of these triangles. Prove that

(a) $R_1^2+R_2^2-R^2$ is positive, negative, or zero according as angle A is acute, obtuse, or right-angled.

(b) $r_1^2+r_2^2-r^2$ is positive, negative, or zero according as angle A is obtuse, acute, or right-angled.

Solution by Michel Bataille, Rouen, France.

(a) Let $\Delta = R_1^2 + R_2^2 - R^2$. Since ABH and ACH are right triangles, we have $R_1 = \frac{1}{2}c = R\sin C$ and $R_2 = \frac{1}{2}b = R\sin B$ (using the familiar notation for $\triangle ABC$). Then

$$\begin{aligned}\Delta &= R^2(\sin^2 C + \sin^2 B - 1) = R^2(\sin^2 B - \cos^2 C)\\ &= R^2\left(\sin B - \sin(\tfrac{\pi}{2} - C)\right)\left(\sin B + \sin(\tfrac{\pi}{2} - C)\right).\end{aligned}$$

Using the identity $(\sin\alpha - \sin\beta)(\sin\alpha + \sin\beta) = \sin(\alpha-\beta)\sin(\alpha+\beta)$, we obtain

$$\Delta = R^2\sin\left(B + C - \tfrac{\pi}{2}\right)\sin\left(B - C + \tfrac{\pi}{2}\right) = R^2\cos A\cos(B-C).$$

Since A is the largest angle of $\triangle ABC$, angles B and C are acute; hence, $\cos(B-C) > 0$. Thus, Δ has the same sign as $\cos A$, and the result follows.

(b) Let $\delta = r_1^2 + r_2^2 - r^2$ and $h = AH$. Since the inradius equals the semiperimeter minus the hypotenuse for right triangles, we calculate

$$\delta = \tfrac{1}{4}(h + HB - c)^2 + \tfrac{1}{4}(h + HC - b)^2 - r^2.$$

Since H is between B and C, we have $HB+HC = a$. Also, $HB = c\cos B$, $HC = b\cos C$, and $2h^2 + HB^2 + HC^2 = b^2 + c^2$; hence,

$$\delta = \tfrac{1}{2}b^2(1-\cos C) + \tfrac{1}{2}c^2(1-\cos B) - \tfrac{1}{2}h(b+c-a) - r^2.$$

Noting that $h = \dfrac{bc\sin A}{a} = \dfrac{bc}{2R}$ and using the Law of Sines, we obtain

$$\begin{aligned}\delta = {}& 2R^2\sin^2 B(1-\cos C) + 2R^2\sin^2 C(1-\cos B)\\ & - 2R^2\sin B\sin C(\sin B + \sin C - \sin A) - r^2.\end{aligned}$$

Now, using the usual half-angle formulas, along with the identity

$$\sin B + \sin C - \sin A = 4\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$$

and the known relation $r = 4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$, we obtain

$$\begin{aligned}\delta &= 16R^2\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C\left(\cos^2\tfrac{1}{2}B+\cos^2\tfrac{1}{2}C\right.\\ &\qquad\left.-\ 2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C-\sin^2\tfrac{1}{2}A\right)\\ &= 8R^2\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C\left(1+\cos A+\cos B+\cos C\right.\\ &\qquad\left.-\ 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\right)\\ &= 8R^2\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C\big(1+\cos A+\cos B+\cos C\\ &\qquad-(\sin A+\sin B+\sin C)\big)\,.\end{aligned}$$

Thus, δ has the same sign as

$$\delta' = (1+\cos A-\sin A)+(\cos B+\cos C)-(\sin B+\sin C)\,.$$

But

$$\begin{aligned}\delta' &= 2\cos\tfrac{1}{2}A\left(\cos\tfrac{1}{2}A-\sin\tfrac{1}{2}A\right)+2\cos\left(\tfrac{1}{2}(B-C)\right)\left(\sin\tfrac{1}{2}A-\cos\tfrac{1}{2}A\right)\\ &= 4\left(\sin\tfrac{1}{2}A-\cos\tfrac{1}{2}A\right)\sin\left(\tfrac{\pi}{4}-\tfrac{1}{2}C\right)\sin\left(\tfrac{\pi}{4}-\tfrac{1}{2}B\right)\end{aligned}$$

has the same sign as $\sin\frac{1}{2}A-\cos\frac{1}{2}A$. The result follows.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN G. HEUVER, Grande Prairie, AB; JOEL SCHLOSBERG, Bayside, NY, USA (part (a) *only*)*; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

In part (a) *of the solution above, Bataille uses the trigonometric identity*

$$(\sin\alpha-\sin\beta)(\sin\alpha+\sin\beta) = \sin(\alpha-\beta)\sin(\alpha+\beta)\,,$$

which is new to this editor. Perhaps we should not tell our students about this one, lest they jump to the conclusion that there is a universal distributive law $\sin(\alpha+\beta)=\sin\alpha+\sin\beta$ *at work here. Ed Barbeau has made similar observations about this identity in* Fallacies, Flaws, and Flimflam, *Mathematical Association of America, 2000, pages 32–33.*

Three of the solvers deduced that δ' (in the notation of the featured solution above) satisfies $\delta' = (2R+r-s)/R$ and then obtained the desired connection between the size of angle A and the sign of δ' by a suitable reference: Heuver to page 232 of [D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989]*; Janous to item 11.27 in* [O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969]*; and Romero to problem 1088* [1985 : 289; 1987 : 124–125].

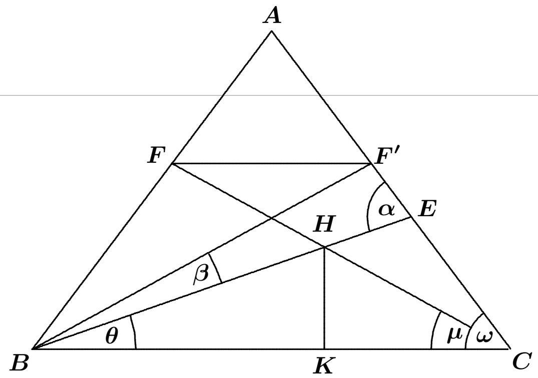
3128. *Proposed by K.R.S. Sastry, Bangalore, India.*

In triangle ABC, we have $AB = AC = 5$, $BC = 6$. Let E be a point on AC and F a point on AB such that $BE = CF$, $\angle EBC \neq \angle FCB$, and $\sin\theta = 5/13$, where $\theta = \angle EBC$. Let H be the point of intersection of BE and CF, and let K be the point on BC such that $HK \perp BC$.

Find the length of HK.

Solution by Roy Barbara, University of Beirut, Beirut, Lebanon.

Let F' be the mirror image of F in the altitude to BC through A. Let $\mu = \angle FCB$, $\omega = \angle ACB$, $\alpha = \angle AEB$, and $\beta = \angle EBF'$. Since $\theta \neq \mu$, we have $F' \neq E$. Note that $\triangle BEF'$ is isosceles, since $BF' = CF = BE$.



From $\sin\theta = 5/13$ and $\cos\omega = 3/5$, we obtain $\tan\theta = 5/12$ and $\tan\omega = 4/3$. We have $\alpha = \theta + \omega$. Thus, $\tan\alpha = \tan(\theta + \omega) = 63/16$. Since $\tan\alpha > 0$, we see that α is an acute angle. Thus, E lies between F' and C (since $\triangle BEF'$ is isosceles). Hence, $\mu = \angle FCB = \angle F'BC = \beta + \theta$.

Next, since $\beta + 2\alpha = \pi$ (summing the angles in $\triangle BEF'$), we calculate

$$\tan\beta \;=\; \tan(\pi - 2\alpha) \;=\; -\tan(2\alpha) \;=\; \frac{2016}{3713}\,.$$

We then deduce that

$$\tan\mu \;=\; \tan(\beta + \theta) \;=\; \frac{253}{204}\,.$$

Since $\tan\theta = \dfrac{HK}{BK}$, $\tan\mu = \dfrac{HK}{KC}$, and $BK + KC = BC = 6$, we obtain

$$HK \;=\; \frac{6\tan\theta\tan\mu}{\tan\theta + \tan\mu} \;=\; \frac{1265}{676}\,.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. Our solvers found many different ways to get to the solution. Two submissions were incorrect.

Konečný commented that he expected a "nicer" answer and asked what the value of $\tan\theta$ *would have to be to get* $HK = 2$*. This editor asks whether there are any pairs of "nice" numbers for* HK *and* $\tan\theta$*, leaving the meaning of "nice" to our readers.*

3129. [2006 : 172, 174] *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, the adjacent internal trisectors of the angles B and C meet at the point P, and the adjacent internal trisectors of the angles A and C meet at the point Q.

Characterize those triangles in which $AQ + BP = AB$.

Solution by Roy Barbara, University of Beirut, Beirut, Lebanon, modified by the editor.

The triangles in which $AQ + BP = AB$ are precisely those with a right angle at C.

Denote by 3α, 3β, and 3γ the angles at A, B, and C, respectively, and by R the circumradius of $\triangle ABC$. Recall that for any real number θ,

$$\begin{aligned} \sin 3\theta &= \sin\theta(3 - 4\sin^2\theta) = \sin\theta(1 + 2\cos 2\theta) \\ &= 4\sin\theta\sin\left(\frac{\pi}{3} - \theta\right)\sin\left(\frac{\pi}{3} + \theta\right) . \end{aligned}$$

These identities will be used without comment.

We have

$$\frac{AB}{\sin\gamma} = \frac{2R\sin 3\gamma}{\sin\gamma} = 2R(1 + 2\cos 2\gamma) . \tag{1}$$

The Law of Sines in $\triangle AQC$ gives

$$\begin{aligned} \frac{AQ}{\sin\gamma} &= \frac{AC}{\sin(\alpha + \gamma)} = \frac{AC}{\sin\left(\frac{\pi}{3} - \beta\right)} = \frac{2R\sin 3\beta}{\sin\left(\frac{\pi}{3} - \beta\right)} \\ &= 8R\sin\beta\sin\left(\tfrac{\pi}{3} + \beta\right) = 4R\Big(\cos\tfrac{\pi}{3} - \cos\left(\tfrac{\pi}{3} + 2\beta\right)\Big) \\ &= 2R\Big(1 - 2\cos\left(\tfrac{\pi}{3} + 2\beta\right)\Big) . \end{aligned} \tag{2}$$

Similarly,

$$\frac{BP}{\sin\gamma} = 2R\Big(1 - 2\cos\left(\tfrac{\pi}{3} + 2\alpha\right)\Big) . \tag{3}$$

Using (1), (2), and (3), we see that $AQ + BP = AB$ if and only if

$$1 - 2\cos\left(\tfrac{\pi}{3} + 2\beta\right) + 1 - 2\cos\left(\tfrac{\pi}{3} + 2\alpha\right) = 1 + 2\cos 2\gamma ;$$

that is,

$$\cos 2\gamma - \tfrac{1}{2} = -\cos\left(\tfrac{\pi}{3} + 2\alpha\right) - \cos\left(\tfrac{\pi}{3} + 2\beta\right) . \tag{4}$$

Now we rewrite both sides of (4):

$$\begin{aligned} \cos 2\gamma - \tfrac{1}{2} &= \cos 2\gamma + \cos\tfrac{2\pi}{3} = 2\cos\left(\tfrac{\pi}{3} - \gamma\right)\cos\left(\tfrac{\pi}{3} + \gamma\right) , \\ &-\cos\left(\tfrac{\pi}{3} + 2\alpha\right) - \cos\left(\tfrac{\pi}{3} + 2\beta\right) \\ &= -2\cos(\alpha - \beta)\cos\left(\tfrac{\pi}{3} + \alpha + \beta\right) \\ &= -2\cos(\alpha - \beta)\cos\left(\tfrac{2\pi}{3} - \gamma\right) = 2\cos(\alpha - \beta)\cos\left(\tfrac{\pi}{3} + \gamma\right) . \end{aligned}$$

Using these expressions in(4), we see that $AQ + BP = AB$ if and only if

$$\cos\left(\tfrac{\pi}{3} - \gamma\right)\cos\left(\tfrac{\pi}{3} + \gamma\right) = \cos(\alpha - \beta)\cos\left(\tfrac{\pi}{3} + \gamma\right) . \tag{5}$$

If there is a right angle at C (that is, if $\gamma = \frac{\pi}{6}$), then both sides of equation (5) are zero. If C is not a right angle ($\gamma \neq \frac{\pi}{6}$), then (5) reduces to

$\cos|\alpha-\beta| = \cos\left(\frac{\pi}{3}-\gamma\right)$. In this case (5) is never satisfied, because the cosine function is injective on $[0,\pi]$ and $0 \le |\alpha-\beta| < \alpha+\beta = \frac{\pi}{3}-\gamma < \pi$.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3130. [2006 : 172, 174] *Proposed by Michel Bataille, Rouen, France.*

Let A, B, C be the angles of a triangle. Show that

$$\left(\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C\right)\left(\csc\tfrac{1}{2}A+\csc\tfrac{1}{2}B+\csc\tfrac{1}{2}C\right) - \left(\cot\tfrac{1}{2}A+\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C\right) \ \ge\ 6\sqrt{3}\,.$$

Essentially the same proof by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; D.M. Milošević, Pranjani, Yugoslavia; and D.J. Smeenk, Zaltbommel, the Netherlands.

Since $A+B+C=\pi$, and $0 < A,\ B,\ C < \pi$, we have

$$\begin{aligned} 0 &= \cos\tfrac{\pi}{2} = \cos\left(\tfrac{1}{2}(A+B+C)\right) \\ &= \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C - \cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C \\ &\qquad - \sin\tfrac{1}{2}A\cos\tfrac{1}{2}B\sin\tfrac{1}{2}C - \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C\,. \end{aligned}$$

We divide by $\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$ and re-arrange the terms to obtain

$$\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A+\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C\,.$$

Since $\cot\frac{1}{2}A$, $\cot\frac{1}{2}B$, and $\cot\frac{1}{2}C$ are each positive, we apply the AM–GM Inequality to get

$$\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C \ \ge\ 3\sqrt[3]{\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C}\,,$$

which implies that

$$\sqrt[3]{\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C} \ \ge\ \sqrt{3}\,. \tag{1}$$

By another application of the AM–GM Inequality, we have

$$\begin{aligned} &\left(\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C\right)\left(\csc\tfrac{1}{2}A+\csc\tfrac{1}{2}B+\csc\tfrac{1}{2}C\right) \\ &\qquad - \left(\cot\tfrac{1}{2}A+\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C\right) \\ &= \frac{\cos\frac{1}{2}A+\cos\frac{1}{2}C}{\sin\frac{1}{2}B} + \frac{\cos\frac{1}{2}B+\cos\frac{1}{2}C}{\sin\frac{1}{2}A} + \frac{\cos\frac{1}{2}A+\cos\frac{1}{2}B}{\sin\frac{1}{2}C} \\ &\ge 6\sqrt[3]{\left(\frac{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}\right)^2} = 6\sqrt[3]{\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C} \ \ge\ 6\sqrt{3}\,, \end{aligned}$$

where we have used (1) to obtain the last inequality. Equality holds if and only if $\cot\frac{1}{2}A = \cot\frac{1}{2}B = \cot\frac{1}{2}C$. Since $\cot t$ is injective on $\left(0, \frac{\pi}{2}\right)$, we see that equality holds if and only if $\triangle ABC$ is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN G. HEUVER, Grande Prairie, AB; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Janous showed that the proposed inequality is a special case of more general inequalities for functions f, $g : (0,\pi) \to \mathbb{R}^+$ satisfying $\prod_{\text{cyclic}} f(A) \geq \lambda$ and either $\prod_{\text{cyclic}} g(A) \geq \mu$ or $\prod_{\text{cyclic}} f(A)g(A) \geq \nu$. In the first case,

$$\Big(\sum_{\text{cyclic}} f(A)\Big)\Big(\sum_{\text{cyclic}} g(A)\Big) - \sum_{\text{cyclic}} f(A)g(A) \;\geq\; 6(\lambda\mu)^{1/3}\,,$$

and in the second case,

$$\Big(\sum_{\text{cyclic}} f(A)\Big)\Big(\sum_{\text{cyclic}} g(A)\Big) - \sum_{\text{cyclic}} f(A)g(A) \;\geq\; 6\nu^{1/3}\,.$$

In both cases, equality holds for equilateral triangles. Janous also noted that the application of inequalities 2.42, 2.32, 2.12, 2.28 in [1] yields more general inequalities of the form

$$\Big(\sum_{\text{cyclic}} (\cos(A/2))^p\Big)\Big(\sum_{\text{cyclic}} (\csc(A/2))^p\Big) - \sum_{\text{cyclic}} (\cot(A/2))^p \;\geq\; 6\cdot 3^{p/2}$$

for any positive real number p.

References

[1] O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969.

3131. [2006 : 172, 175] *Proposed by Michel Bataille, Rouen, France.*

The normal at M to a conic with focus F meets the focal axis at N. Let H and K be points on MF such that $HN \perp MF$ and $KN \perp MN$. If $\frac{1}{HN} = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$ and $NK = \sqrt{ab}$ (where $a > b > 0$), show that $KI = (a+b)/2$ for some significant point I on MN.

I. *Solution by D.J. Smeenk, Zaltbommel, the Netherlands.*

One possible position for I is the point where the line MN intersects the line perpendicular to MK at K: the right triangles HNK and NKI are similar; whence, $IK/KN = KN/NH$, and

$$KI \;=\; \frac{\sqrt{ab}\cdot\sqrt{ab}}{2ab/(a+b)} \;=\; \frac{a+b}{2}\,.$$

[*Ed.*: We shall see below that the point I is more "significant" than can be seen from Smeenk's solution. It is, in fact, the centre of curvature of the conic at the point M (that is, the centre of the circle whose curvature is the same as the conic's at their shared point M).]

II. *Solution for the case of an ellipse by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

We assume that the conic is the ellipse $x^2/\alpha^2 + y^2/\beta^2 = 1$; the focus F is the point $(c, 0)$, where c satisfies $c^2 + \beta^2 = \alpha^2$. For some value of t, $0 < t \leq \pi/2$, let M be the point $(\alpha \cos t, \beta \sin t)$. Using standard calculus, one determines the radius of curvature at M to be $k^3/(\alpha\beta)$, where

$$k^2 \;=\; \alpha^2 \sin^2 t + \beta^2 \cos^2 t\,.$$

The slope of the tangent at M is $-\beta \cos t/\alpha \sin t$, so that the slope of the normal there must be $\alpha \sin t/\beta \cos t$. Therefore, MN has the equation $y - \beta \sin t = \dfrac{\alpha \sin t}{\beta \cos t}(x - \alpha \cos t)$, or

$$(\beta \cos t)y - (\alpha \sin t)x + c^2 \sin t \cos t \;=\; 0\,.$$

This line meets the x–axis at $N\left(\dfrac{c^2 \cos t}{\alpha}, 0\right)$; whence,

$$MN^2 \;=\; \beta^2 \sin^2 t + \left(\frac{\beta^2 \cos t}{\alpha}\right)^2 \;=\; \frac{\beta^2 k^2}{\alpha^2}\,. \tag{1}$$

Since the equation of the line MF is $y = -(x - c)\beta \sin t/(c - \alpha \cos t)$, the distance from N to MF is

$$NH \;=\; \frac{\left|\dfrac{\beta c^2 \sin t \cos t}{a} - \beta c \sin t\right|}{\sqrt{\beta^2 \sin^2 t + (c - \alpha \cos t)^2}}\,.$$

The square-root term in the denominator simplifies as follows:

$$\begin{aligned}
\beta^2 \sin^2 t + (c - \alpha \cos t)^2 &= \beta^2 \sin^2 t + c^2 - 2\alpha c \cos t + \alpha^2 \cos^2 t \\
&= \beta^2 \sin^2 t + c^2 - 2\alpha c \cos t + (\beta^2 + c^2) \cos^2 t \\
&= \alpha^2 - 2\alpha c \cos t + c^2 \cos^2 t \;=\; (\alpha - c \cos t)^2\,;
\end{aligned}$$

thus, $NH = \beta c \sin t/\alpha$, and

$$\begin{aligned}
MH^2 &= MN^2 - NH^2 \;=\; \frac{\beta^2 k^2}{\alpha^2} - \frac{\beta^2 c^2 \sin^2 t}{\alpha^2} \\
&= \frac{\beta^2(\alpha^2 \sin^2 t + (\alpha^2 - c^2) \cos^2 t) - \beta^2 c^2 \sin^2 t}{\alpha^2} \\
&= \frac{\beta^2(\alpha^2 - c^2 \cos^2 t - c^2 \sin^2 t)}{\alpha^2} \;=\; \frac{\beta^4}{\alpha^2}\,.
\end{aligned}$$

Hence,

$$MH \;=\; \beta^2/\alpha \tag{2}$$

(which, incidentally, is both the latus rectum of the ellipse and the radius of curvature at $(\alpha, 0)$).

From the similar right triangles MNK and MHN, we see that $MK/MN = MN/MH$. Combining this proportion with equations (1) and (2), we see that

$$MK = \frac{MN^2}{MH} = \frac{k^2}{\alpha}.$$

Finally, from the similar right triangles IMK and NMH,

$$IM = \frac{MN \cdot MK}{MH} = \frac{k^3}{\alpha\beta},$$

so that IM is the radius of curvature at M, as claimed.

Complete solutions came from APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; and the proposer.

The solutions of Bataille and Demis used a general form for a conic that produces the centre of curvature for an arbitrary conic—Bataille's is based on the latus rectum while Demis' is based on the eccentricity. We have presented Woo's solution even though it is valid only for an ellipse, because it has the virtue of using familiar formulas. The corresponding formulas for the hyperbola and the parabola may be obtained easily.

This problem provides a simple construction for the centre of curvature of a conic at any point M that is not on the major axis: Join M to the foci F and F'; call N the point where the bisector of $\angle FMF'$ meets FF'. (This is the normal to the conic at M; in the case of a parabola, MF' is taken to be the line through M that is parallel to the axis.) Define K to be the point where the perpendicular to MN at N meets MF; then I is the point where the perpendicular to MF at K meets the normal MN. Clearly, the numbers a and b that appear in the problem are a function of M; they are only indirectly related to the lengths of the semi-major and semi-minor axes that are usually denoted by these letters. The notation seems to have been chosen by Bataille for its nice relationship to the construction of the point I.

3132. [2006 : 172, 174] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $F(n)$ be the number of ones in the binary expression of the positive integer n. For example,

$$F(5) = F\left(101_{(2)}\right) = 2,$$
$$F(15) = F\left(1111_{(2)}\right) = 4.$$

Let $S_k = \sum_{n=1}^{\infty} \frac{F^k(n)}{n(n+1)}$, where $F^k(n)$ is defined recursively by $F^1 = F$ and $F^k = F \circ F^{k-1}$ for $k \geq 2$.

(a) Prove that $S_1 = 2\ln 2$.

(b) Prove that $\frac{18}{5}\ln 2 - \frac{1}{15} \leq S_2 \leq 4\ln 2$.

(c) Prove that $\frac{218}{25}\ln 2 - \frac{7}{25} \leq S_3 \leq 11\ln 2$.

(d)★ Compute S_k.

[*Ed:* In this problem, the expression $F^k(n)$ means $\left(F(n)\right)^k$, rather than the k^{th} iterate of F, as stated above.]

Solution to part (a) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For any positive integer n, denote the binary representation of n by $\sum\limits_{j=0}^{\infty} z_j(n)2^j$. Then $F(n) = \sum\limits_{j=0}^{\infty} z_j(n)$. Now

$$\begin{aligned} S_1 &= \sum_{n=1}^{\infty} \frac{F(n)}{n(n+1)} = \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{z_j(n)}{n(n+1)} = \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} \frac{z_j(n)}{n(n+1)} \\ &= \sum_{j=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{u=0}^{2^j-1} \frac{1}{(\nu 2^{j+1}+2^j+u)(\nu 2^{j+1}+2^j+u+1)} \\ &= \sum_{j=0}^{\infty}\sum_{\nu=0}^{\infty} \left(\frac{1}{\nu 2^{j+1}+2^j} - \frac{1}{\nu 2^{j+1}+2^j+2^j}\right) \\ &= \sum_{j=0}^{\infty}\sum_{\nu=0}^{\infty} \left(\frac{1}{(2\nu+1)2^j} - \frac{1}{(2\nu+2)2^j}\right) \\ &= \sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{\nu=0}^{\infty} \left(\frac{1}{2\nu+1} - \frac{1}{2\nu+2}\right) = \sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 2\ln 2\,. \end{aligned}$$

Solution to parts (b) *and* (c) *by the proposer.*

We have $F(2m) = F(m)$ and $F(2m+1) = 1 + F(2m) = 1 + F(m)$. Let $a_k \leq S_k \leq b_k$. Then

$$\begin{aligned} S_k &= \sum_{n=1}^{\infty} \frac{F^k(n)}{n(n+1)} = \sum_{m=0}^{\infty} \frac{\big(1+F(m)\big)^k}{(2m+1)(2m+2)} + \sum_{m=1}^{\infty} \frac{F^k(m)}{2m(2m+1)} \\ &= \sum_{m=0}^{\infty} \frac{1}{(2m+1)(2m+2)} + \sum_{m=0}^{\infty} \frac{\sum\limits_{p=1}^{k-1}\binom{k}{p}F^p(m)}{(2m+1)(2m+2)} \\ &\qquad + \sum_{m=0}^{\infty} \frac{F^k(m)}{(2m+1)(2m+2)} + \sum_{m=1}^{\infty} \frac{F^k(2m)}{2m(2m+1)} \\ &= \ln 2 + \sum_{m=1}^{\infty} \frac{F^k(m)}{2(2m+1)}\left(\frac{1}{m+1}+\frac{1}{m}\right) + \frac{\sum\limits_{p=1}^{k-1}\binom{k}{p}F^p(m)}{(2m+1)(2m+2)} \\ &= \ln 2 + \frac{1}{2}S_k + \sum_{m=1}^{\infty} \frac{\sum\limits_{p=1}^{k-1}\binom{k}{p}F^p(m)}{(2m+1)(2m+2)}\,. \end{aligned}$$

Hence,

$$S_k = 2\ln 2 + 2\sum_{p=1}^{k-1}\binom{k}{p}\sum_{m=1}^{\infty}\frac{F^p(m)}{(2m+1)(2m+2)}\,. \tag{1}$$

Since $\dfrac{1}{(2m+1)(2m+2)} \le \dfrac{1}{4m(m+1)}$, from (1) we have

$$S_k \;\le\; 2\ln 2 + \frac{1}{2}\sum_{p=1}^{k-1}\binom{k}{p}\sum_{m=1}^{\infty}\frac{F^p(m)}{m(m+1)} \;\le\; 2\ln 2 + \frac{1}{2}\sum_{p=1}^{k-1}\binom{k}{p}b_p\,.$$

From part (a), we have $b_1 = 2\ln 2$ and, for $k = 2$, we conclude that

$$S_2 \;\le\; 2\ln 2 + \frac{1}{2}\binom{2}{1}b_1 \;=\; 4\ln 2\,;$$

thus, $b_2 = 4\ln 2$. For $k = 3$, we have

$$S_3 \;\le\; 2\ln 2 + \frac{1}{2}\left[\binom{3}{1} + \binom{3}{2}\right] \;=\; 11\ln 2\,,$$

which means that $b_3 = 11\ln 2$.

On the other hand, $\dfrac{1}{(2m+1)(2m+2)} \;\ge\; \dfrac{1}{5m(m+1)}$ for $m \ge 2$, and from (1) we have

$$\begin{aligned} S_k &\;\ge\; 2\ln 2 + \frac{2}{5}\sum_{p=1}^{k-1}\binom{k}{p}\sum_{m=2}^{\infty}\frac{F^p(m)}{m(m+1)} + \frac{1}{6}\sum_{p=1}^{k-1}\binom{k}{p} \\ &\;\ge\; 2\ln 2 + \frac{2}{5}\left(\sum_{p=1}^{k-1}\binom{k}{p}\sum_{m=2}^{\infty}\frac{F^p(m)}{m(m+1)} - \frac{1}{2}\sum_{p=1}^{k-1}\binom{k}{p}\right) + \frac{2(2^k-2)}{12} \\ &\;\ge\; 2\ln 2 - \frac{2^k-2}{30} + \frac{2}{5}\sum_{p=1}^{k-1}\binom{k}{p}a_p\,. \end{aligned}$$

For $k = 2$, we conclude from part (a) that

$$S_2 \;\ge\; 2\ln 2 - \frac{1}{15} + \frac{2}{5}\binom{2}{1}a_1 \;=\; \frac{18}{5}\ln 2 - \frac{1}{15}\,.$$

Since $a_2 = \frac{18}{5}\ln 2 - \frac{1}{15}$, we have

$$S_3 \;\ge\; 2\ln 2 - \frac{1}{5} + \frac{2}{5}\left[\binom{3}{1}a_1 + \binom{3}{2}a_2\right] \;\ge\; \frac{118}{25}\ln 2 - \frac{7}{25}\,.$$

Part (a) *also solved by JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer. Part* (d) *remains open.*

3133. [2006 : 172, 175] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be any triangle. Show that

$$\sum_{\text{cyclic}}\frac{1 + 2\sin A - \cos 2A}{8 + 3\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) + \cos\left(\frac{3A}{2}\right)\cos\left(\frac{3(B-C)}{2}\right)} \;\le\; 1\,.$$

Composite of almost identical solutions by Michel Bataille, Rouen, France; and the proposer.

We first make the following observations:

$$1 + 2\sin A - \cos 2A \;=\; 2\sin A + 2\sin^2 A\,, \tag{1}$$

$$\begin{aligned} 2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) &= \cos\left(\frac{A+B-C}{2}\right) + \cos\left(\frac{A+C-B}{2}\right) \\ &= \cos\left(\frac{\pi-2C}{2}\right) + \cos\left(\frac{\pi-2B}{2}\right) \\ &= \sin C + \sin B\,, \end{aligned} \tag{2}$$

$$\begin{aligned} &2\cos\left(\frac{3A}{2}\right)\cos\left(\frac{3(B-C)}{2}\right) \\ &\qquad = \cos\left(\frac{3(A+B-C)}{2}\right) + \cos\left(\frac{3(A+C-B)}{2}\right) \\ &\qquad = \cos\left(\frac{3(\pi-2C)}{2}\right) + \cos\left(\frac{3(\pi-2B)}{2}\right) \\ &\qquad = -\sin 3C - \sin 3B\,. \end{aligned} \tag{3}$$

Using (2), (3), and the formula $\sin 3x = 3\sin x - 4\sin^3 x$, we get

$$\begin{aligned} &8 + 3\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) + \cos\left(\frac{3A}{2}\right)\cos\left(\frac{3(B-C)}{2}\right) \\ &\quad = 8 + \tfrac{3}{2}(\sin B + \sin C) - \tfrac{1}{2}(\sin 3B + \sin 3C) \\ &\quad = 2(4 + \sin^3 B + \sin^3 C) \;\ge\; 2(3 + \sin^3 A + \sin^3 B + \sin^3 C)\,. \end{aligned} \tag{4}$$

Let L denote the left side of the inequality in the problem statement. Using (1) and (4), we obtain

$$L \;\le\; \sum_{\text{cyclic}} \frac{2\sin A + 2\sin^2 A}{2(3 + \sin^3 A + \sin^3 B + \sin^3 C)} \;=\; \frac{\sum\limits_{\text{cyclic}} (\sin A + \sin^2 A)}{\sum\limits_{\text{cyclic}} (1 + \sin^3 A)}\,.$$

Hence, it suffices to show that $\sum\limits_{\text{cyclic}} (\sin A + \sin^2 A) \le \sum\limits_{\text{cyclic}} (1 + \sin^3 A)$, which is equivalent in succession to

$$\begin{aligned} \sum_{\text{cyclic}} (1 + \sin^3 A - \sin A - \sin^2 A) &\;\ge\; 0\,, \\ \sum_{\text{cyclic}} (1 + \sin A)(1 - \sin A)^2 &\;\ge\; 0\,. \end{aligned}$$

Since the last inequality is clearly true, our proof is complete.

Also solved by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

From the last inequality in the proof featured above, it is clear that equality cannot hold in the given inequality. This was explicitly pointed out by Janous, who believes that the best upper bound λ *for* L *is* $\lambda = (90 + 69\sqrt{3})/229 \approx 0.915$, *attained when* $A = B = C = \frac{\pi}{3}$.

3134. [2006 : 173, 175] *Proposed by Mihály Bencze, Brasov, Romania.*

Let O be the circumcentre of $\triangle ABC$. Let D, E, and F be the mid-points of BC, CA, and AB, respectively; let K, M, and N be the mid-points of OA, OB, and OC, respectively. Denote the circumradius, inradius, and semiperimeter of $\triangle ABC$ by R, r, and s, respectively. Prove that

$$2(KD+ME+NF) \;\ge\; R+3r+\frac{s^2+r^2}{2R}\,.$$

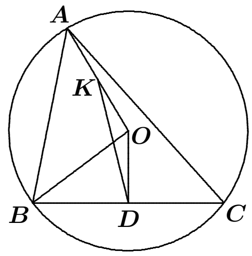
Solution by Michel Bataille, Rouen, France, modified by the editor.

First, we prove that

$$\frac{KD^2}{R^2} \;=\; \frac{1}{4}+\cos^2 A+\cos A\cos(B-C)\,. \tag{1}$$

We can assume that B is closer to A than to C.

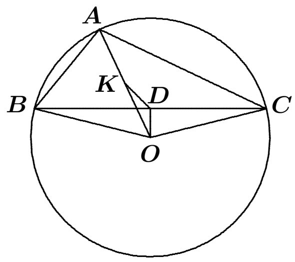
If $\angle A$ is a right angle, then the points D and O coincide and (1) is equivalent to $R=2KO$, which is clearly true.



If $\angle A$ is acute, then

$$\begin{aligned}\angle KOD &= \angle AOB+\angle BOD\\ &= 2C+A\\ &= \pi-(B-C)\end{aligned}$$

and $OD = R\cos A$. The Law of Cosines applied in $\triangle KOD$ yields (1).



If $\angle A$ is obtuse, then

$$\begin{aligned}\angle KOD &= \angle BOD-\angle AOB\\ &= (\pi-A)-2C\\ &= B-C\end{aligned}$$

and $OD = R\cos(\pi-A) = R\cos A$. Again the Law of Cosines in $\triangle KOD$ yields (1).

From (1), we get

$$\begin{aligned}\frac{KD^2}{R^2} &= \left[\cos A+\frac{1}{2}\cos(B-C)\right]^2+\frac{1}{4}\sin^2(B-C)\\ &\ge \left[\cos A+\frac{1}{2}\cos(B-C)\right]^2\,,\end{aligned}$$

and therefore,

$$KD \;\ge\; R\left|\cos A+\tfrac{1}{2}\cos(B-C)\right| \;\ge\; R\left[\cos A+\tfrac{1}{2}\cos(B-C)\right]\,.$$

Similar inequalities hold for ME and NF. Adding these and using the well-known formulas

$$\begin{aligned}\sum_{\text{cyclic}} \cos A &= 1 + \frac{r}{R}\,, \\ \sum_{\text{cyclic}} \cos B \cos C &= \frac{r^2 + s^2 - 4R^2}{4R^2}\,, \\ \text{and} \qquad \sum_{\text{cyclic}} \sin B \sin C &= \frac{r^2 + s^2 + 4Rr}{4R^2}\,,\end{aligned}$$

we obtain

$$\begin{aligned}&2(KD + ME + NF) \\ &\quad\geq\ 2R\left[\sum_{\text{cyclic}} \cos A + \tfrac{1}{2}\sum_{\text{cyclic}} \cos(B - C)\right] \\ &\quad=\ R\left[2\left(1 + \frac{r}{R}\right) + \frac{r^2 + s^2 - 4R^2}{4R^2} + \frac{r^2 + s^2 + 4Rr}{4R^2}\right] \\ &\quad=\ R + 3r + \frac{s^2 + r^2}{2R}\,.\end{aligned}$$

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3135. [2006 : 173, 176], corrected [2006 : 303, 306], corrected again [2006 : 514, 516] *Proposed by Marian Marinescu, Monbonnot, France.*

Let $\mathbb{R}^+$ be the set of non-negative real numbers. For all a, b, $c \in \mathbb{R}^+$, let $H(a,b,c)$ be the set of all functions $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$h(x) \ \geq\ h\big(h(ax)\big) + h(bx) + cx$$

for all $x \in \mathbb{R}^+$. Prove that $H(a,b,c)$ is non-empty if and only if $b \leq 1$ and $4ac \leq (1-b)^2$.

[*Ed:* The version of this problem that was originally printed in Crux [2006 : 173, 176] was revised twice in later issues to correct typographical errors. Above is the final corrected version [2006 : 514, 516], as submitted by the proposer. Unfortunately, it is still not quite right, as can be seen by observing that the zero function $h(x) \equiv 0$ is always in $H(a,b,c)$ if $c = 0$. The stated equivalence also fails in the case where $c > 0$, $a = 0$ and $b = 1$. To avoid dealing with these special cases, we will correct the problem (once more!) by requiring $c > 0$ instead of just $c \geq 0$ and then asking for a proof that $H(a,b,c)$ is non-empty if and only if $b < 1$ and $4ac \leq (1-b)^2$. The proposer's solution is then essentially correct.]

Solution by the proposer, modified by the editor.

First suppose that $b < 1$ and $4ac \leq (1-b)^2$. Let p be any real number such that $ap^2 - (1-b)p + c \leq 0$. For example, choose $p = (1-b)/(2a)$ if $a > 0$ and $p = c/(1-b)$ if $a = 0$. Then the function $h(x) = px$ satisfies the functional inequality in the problem. Thus, $H(a,b,c)$ is non-empty.

Now suppose $H(a,b,c)$ is non-empty. Choose $h \in H(a,b,c)$, and let $r = \inf\{h(x)/x : x > 0\}$. Then $r \geq 0$ and $h(x) \geq rx$ for all $x \geq 0$. Then $h(bx) \geq rbx$ and $h\big(h(ax)\big) \geq rh(ax) \geq r^2ax$ for all $x \geq 0$. Using the given functional inequality (which is satisfied by h), we get $h(x) \geq (ar^2 + br + c)x$ for all $x \geq 0$. Then, by the definition of r, we must have $ar^2 + br + c \leq r$; that is, $ar^2 - (1-b)r + c \leq 0$. Since r is real and non-negative, it follows that $4ac \leq (1-b)^2$ and $b < 1$.

Also solved by ROY BARBARA, University of Beirut, Beirut, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

Most solutions were for the first corrected version of the problem [2006 : 303, 306]. These solutions have been accepted as correct, since they contained the main ideas needed to solve the problem in the form given above.

3136. [2006 : 173, 176] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle with circumcircle Γ; let ℓ be a transversal which meets the line BC at L, the line CA at M, and the line AB at N. Let Γ_1 be the circle through A which is tangent to BC at L, and let Γ_2 and Γ_3 be similarly defined with respect to B and C. Let QR, RP, and PQ be the common chords of Γ and Γ_1, Γ and Γ_2, and Γ and Γ_3, respectively. Prove that AP, BQ, and CR are concurrent.

Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece.

We denote the intersection point of BC and QR by A', of CA and RP by B', and of AB and PQ by C'. Define K to be the point where BQ and AP intersect; we are to prove that K lies also on CR. Also, let us denote the second points of intersection of Γ with Γ_1, Γ_2, Γ_3 by D, E, F, respectively.

Applying the Theorem of Menelaus to triangle ABC with respect to the transversal ℓ, we get

$$\frac{AM}{MC} \cdot \frac{CL}{LB} \cdot \frac{BN}{NA} = -1\,.$$

The power of the point C' with respect to Γ and Γ_3 is $C'A \cdot C'B = C'F \cdot C'C = C'N^2$; whence,

$$\frac{C'N}{C'A} = \frac{C'B}{C'N} = \frac{C'B - C'N}{C'N - C'A}\,.$$

Thus, $(C'N/C'A) \cdot (C'B/C'N) = (C'B - C'N)^2/(C'N - C'A)^2$, or (in terms of oriented line segments)

$$\frac{BC'}{C'A} = -\frac{BN^2}{NA^2}.$$

Similarly,

$$\frac{CA'}{A'B} = -\frac{CL^2}{LB^2} \quad \text{and} \quad \frac{AB'}{B'C} = -\frac{AM^2}{MC^2}.$$

Therefore,

$$\frac{AB'}{B'C} \cdot \frac{CA'}{A'B} \cdot \frac{BC'}{C'A} = -\frac{AM^2}{MC^2} \cdot \frac{CL^2}{LB^2} \cdot \frac{BN^2}{NA^2} = -1,$$

and, consequently, the points A', B', and C' are collinear. The triangles APB' and BQA' are therefore perspective from C', so that, by Desargues' Theorem, we deduce that the points $K = AP \cap BQ$, $R = PB' \cap QA'$, and $C = B'A \cap A'B$ are collinear. Thus, K is the common point of the lines CR, BQ, and AP, as desired.

Comment. There are other interesting incidences to be discovered in this rich configuration. For example, C' lies on the line joining the points $AP \cap BC$ and $BQ \cap AC$. Also, C' belongs to the Pascal line defined by the hexagon $ABECFD$.

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