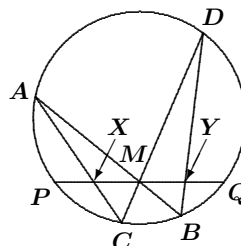


Butterfly Metamorphosis

Andy Liu

The Butterfly Theorem is a result which has acquired cult status. For two important surveys, see [1] and [5]. Much of this is later repeated in [3]. The setting of the Butterfly Theorem involves three concurrent chords in a circle.

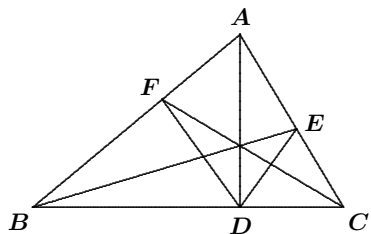
Butterfly Theorem. Let PQ , AB , and CD be three chords of a circle concurrent at a point M , with A and D on one side of PQ and B and C on the other side. If $PM = QM$, then $XM = YM$, where X and Y are the points of intersection of PQ with AC and BD , respectively.



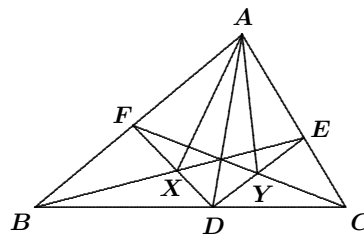
Our metamorphosis changes the setting to three concurrent cevians in a triangle. We will use techniques developed below to give a simple proof of the Butterfly Theorem.

Theorem. Let AD , BE , and CF be three concurrent cevians in $\triangle ABC$.

- (a) *First Metamorphosis:* If $\angle ADB = \angle ADC$, then $\angle ADF = \angle ADE$.
- (b) *Second Metamorphosis:* If $\angle DAB = \angle DAC$, then $\angle DAX = \angle DAY$, where X is the point of intersection of FD and BE , and Y is the point of intersection of ED and CF .



(a)



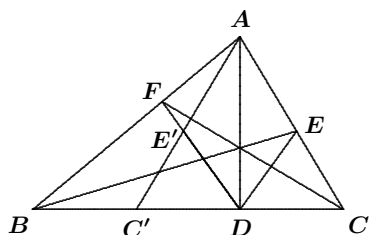
(b)

The condition $\angle ADB = \angle ADC$ in part (a) is, of course, just a clumsy way of saying that AD is an altitude. However, stating it this way highlights the relationship of this result to the Butterfly Theorem. This was, for instance, not observed in [4].

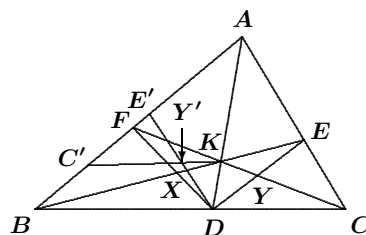
Proof: Our approach here is by symmetry.

(a) We fold $\angle BDC$ along its bisector AD , so that the image C' of C lies on BD and the image E' of E lies on AC' . The desired result is now

equivalent to D , E' , and F being collinear. By Ceva's Theorem, we have $\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = 1$. Since $\frac{\overline{BD}}{\overline{DC}} = -\frac{\overline{BD}}{\overline{DC}'}$ while $\frac{\overline{CE}}{\overline{EA}} = \frac{\overline{C'E'}}{\overline{E'A}}$, we have $\frac{\overline{BD}}{\overline{DC}'} \cdot \frac{\overline{C'E'}}{\overline{E'A}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1$. By the converse of Menelaus' Theorem, D , E' , and F are indeed collinear.



(a)



(b)

(b) Let K be the point of concurrency of AD , BE , and CF . This time, we fold $\angle BAC$ along its bisector AD , so that the image C' of C and the image E' of E lie on AB , while the image Y' of Y is the point of intersection of DE' and KC' . The desired result is now equivalent to A , X , and Y' being collinear. By Menelaus' Theorem, we have

$$\begin{aligned} \frac{\overline{EK}}{\overline{KB}} \cdot \frac{\overline{BC}}{\overline{CD}} \cdot \frac{\overline{DY}}{\overline{YE}} &= -1, & \frac{\overline{BK}}{\overline{KE}} \cdot \frac{\overline{EA}}{\overline{AC}} \cdot \frac{\overline{CD}}{\overline{DB}} &= -1, \\ \frac{\overline{CK}}{\overline{KF}} \cdot \frac{\overline{FX}}{\overline{XD}} \cdot \frac{\overline{DB}}{\overline{BC}} &= -1, & \frac{\overline{FK}}{\overline{KC}} \cdot \frac{\overline{CD}}{\overline{DB}} \cdot \frac{\overline{BA}}{\overline{AF}} &= -1. \end{aligned}$$

Multiplication and cancellation yields

$$\frac{\overline{FX}}{\overline{XD}} \cdot \frac{\overline{DY}}{\overline{YE}} \cdot \frac{\overline{EA}}{\overline{AC}} \cdot \frac{\overline{CD}}{\overline{DB}} \cdot \frac{\overline{BA}}{\overline{AF}} = 1.$$

Because AD bisects $\angle CAB$, we have $\frac{BA}{AC} = \frac{BD}{DC}$. It follows that

$$\frac{\overline{EA}}{\overline{AC}} \cdot \frac{\overline{CD}}{\overline{DB}} \cdot \frac{\overline{BA}}{\overline{AF}} = -\frac{\overline{E'A}}{\overline{AF}},$$

so that $\frac{\overline{FX}}{\overline{XD}} \cdot \frac{\overline{DY}}{\overline{YE}} \cdot \frac{\overline{E'A}}{\overline{AF}} = -1$. By the converse of Menelaus' Theorem, A , X , and Y' are collinear. ■

The First Metamorphosis later became Problem 5 of the 1994 Canadian Mathematical Olympiad. Our approach is different from all known proofs.

The Second Metamorphosis appeared as Problem 6 in the Spring 2006 Senior Advanced Level Paper of the International Mathematics of the Towns. Our approach is different from the official solution provided.

The approaches we have used so far provide a plausible motivation to perhaps the simplest proof of the Butterfly Theorem. We give the argument in [2] (repeated in [1]) in this light.

We fold PQ along its perpendicular bisector so that the image D' of D is the point of intersection of the circle and the line through D parallel to PQ . What we want to prove is that X coincides with the image Y' of Y . This will follow if we can prove that triangles DMY and $D'MX$ are congruent.

We have $DM = D'M$. Hence,

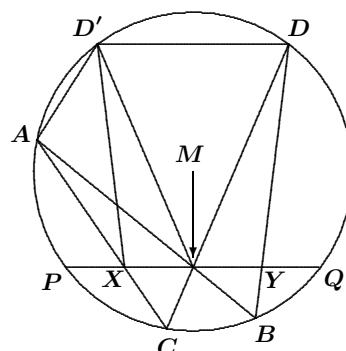
$$\angle DMY = \angle MDD' = \angle MD'D = \angle D'MX.$$

We will now prove that $\angle MDY = \angle MD'X$. Since $ACBD$ is a cyclic quadrilateral, $\angle MDY = \angle CAB$. We will have $\angle CAB = \angle MD'X$ if we can prove that $AD'MX$ is also a cyclic quadrilateral. Since $ACDD'$ is cyclic, $\angle D'AX + \angle MDD' = 180^\circ$. However, we have already proved that $\angle MDD' = \angle D'MX$, so that $\angle D'AX + \angle D'MX = 180^\circ$ too. Hence, $AD'MX$ is indeed cyclic, and it follows that $MX = MY$.

Acknowledgment. The author would like to thank the anonymous referee for some critical comments and useful references.

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