THE OLYMPIAD CORNER

No. 261

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We begin this number with the problems of the XX Olimpiadi Italiane della Matematica. My thanks go to Christopher Small, Canadian Team Leader to the IMO in Athens, Greece for collecting them for our use.

XX OLIMPIADI ITALIANE DELLA MATEMATICA
Cesenatico, 7 May 2004

1. Reading the temperatures in Cesenatico for the months of December and January, Stefano notices an odd feature: on each day in that period, except for the first and the last, the lowest temperature was the sum of the lowest temperatures on the day before and the day after.

The lowest temperature was 5°C on December 3 and 2°C on January 31. Find the lowest temperature on December 25.

2. Let r and s be two parallel lines in the plane, and P and Q two points such that $P \in r$ and $Q \in s$. Consider circles C_P and C_Q such that C_P is tangent to r at P, C_Q is tangent to s at Q, and C_P and C_Q are tangent externally to each other at some point, say T. Find the locus of T when (C_P, C_Q) varies over all pairs of circles with the given properties.

3. (a) Determine whether the number 2005^{2004} can be written as the sum of the squares of two positive integers.

(b) Determine whether the number 2004^{2005} can be written as the sum of the squares of two positive integers.

4. Antonio and Bernardo play the following game: In the beginning there are two piles of tokens, one with m tokens and the other with n tokens. Each player in turn chooses one of the following moves:

- remove one token from one pile;
- remove one token from each of the two piles;
- move one token from one pile to the other.

The player with no possible moves loses.

Antonio always moves first. Depending on m and n, determine whether one of the two players has a winning strategy, and, if so, show who is the winning player.

5. Determine whether the following statement is true or false: For every sequence $x_1, x_2, x_3, \ldots$ of non-negative real numbers, there exist two sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ of non-negative real numbers such that

- $x_n = a_n + b_n$ for each n;
- $a_1 + \cdots + a_m \leq m$ for infinitely many m; and
- $b_1 + \cdots + b_l \leq \ell$ for infinitely many ℓ.

6. Let P be a point inside the triangle ABC. Say that the lines AP, BP, and CP meet the sides of ABC at A', B', and C', respectively. Let

$$x = \frac{AP}{PA'}, \quad y = \frac{BP}{PB'}, \quad z = \frac{CP}{PC'}.$$

Prove that $xyz = x + y + z + 2$.

Next we give the First and Second Papers of the Seventeenth Irish Mathematical Olympiad given in May 2004. Thanks again to Christopher Small, Canadian Team Leader to the IMO in Athens, for collecting them for our use.

17th IRISH MATHEMATICAL OLYMPIAD
First Paper — May 8, 2004

1. (a) For which positive integers n does $2n$ divide the sum of the first n positive integers?

(b) Determine, with proof, those positive integers n (if any) which have the property that $2n + 1$ divides the sum of the first n positive integers.

2. Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there is a group A, B, C of three players for which A beat B, B beat C, and C beat A.

3. Let AB be a chord of length 6 of a circle of radius 5 centred at O. Let $PQRS$ denote the square inscribed in the sector OAB such that P is on the radius OA, S is on the radius OB, and Q and R are points on the arc of the circle between A and B. Find the area of $PQRS$.

4. Prove that there are only two real numbers x such that

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) = 720.$$

5. Let $a, b \geq 0$. Prove that

$$\sqrt{2}\left(\sqrt{a(a+b)^3} + b\sqrt{a^2+b^2}\right) \leq 3(a^2+b^2),$$

with equality if and only if $a = b$.

17th IRISH MATHEMATICAL OLYMPIAD
Second Paper — May 8, 2004

1. Determine all pairs of prime numbers (p, q), with $2 \leq p, q < 100$, such that $p + 6$, $p + 10$, $q + 4$, $q + 10$, and $p + q + 1$ are all prime numbers.

2. Let A and B be distinct points on a circle T. Let C be a point distinct from B such that $|AB| = |AC|$ and such that BC is tangent to T at B. Suppose that the bisector of $\angle ABC$ meets AC at a point D inside T. Show that $\angle ABC > 72^\circ$.

3. Suppose n is an integer ≥ 2. Determine the first digit after the decimal point in the decimal expansion of the number $\sqrt[3]{n^3 + 2n^2 + n}$.

4. Define the function m of the three real variables x, y, and z by

$$m(x, y, z) = \max\{x^2, y^2, z^2\} .$$

Determine, with proof, the minimum value of m if x, y, and z vary in $\mathbb{R}$ subject to the restrictions $x + y + z = 0$ and $x^2 + y^2 + z^2 = 1$.

5. Let p and q be distinct primes and let S be a subset of $\{1, 2, \ldots, p - 1\}$. Let $N(S)$ denote the number of solutions of the equation

$$\sum_{i=1}^{q} x_i \equiv 0 \pmod{p} ,$$

where $x_i \in S$, $i = 1, 2, \ldots, q$. Prove that $N(S)$ is a multiple of q.

Our last set of problems is the IMO Squad Selection Problems 2004 from the New Zealand Mathematical Olympiad. Thanks to Christopher Small, Canadian Team Leader to the IMO in Athens, for obtaining them for us.

NEW ZEALAND MATHEMATICAL OLYMPIAD
IMO Squad Selection Problems 2004

1. Let I be the incentre of triangle ABC, and let A', B', and C' be the reflections of I in BC, CA, and AB, respectively. The circle through A', B', and C' passes also through B. Find the angle $\angle ABC$.

2. Two players are taking turns to write integers on the blackboard in the range from 1 to 1000. The first player starts by writing the number 1. If the number a was already written on the board (please note that the numbers written at early stages are not erased), then the next number may be either $a + 1$ or $2a$, provided that the last number does not exceed 1000. The player who writes 1000 wins. Which player, the first or the second, has a winning strategy?

3. For positive x_1, x_2, y_1, y_2, prove the inequality

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} \geq \frac{(x_1+x_2)^2}{y_1+y_2}\,.$$

4. For each positive integer n, let $w(n)$ be the number of positive prime divisors of n. Find the smallest positive integer k such that for all n

$$2^{w(n)} \leq k\sqrt[4]{n}\,.$$

5. Let I be the incentre of triangle ABC. Let points $A_1 \neq A_2$ lie on the line BC, points $B_1 \neq B_2$ lie on the line AC, and points $C_1 \neq C_2$ lie on the line AB so that $AI = A_1I = A_2I$, $BI = B_1I = B_2I$, $CI = C_1I = C_2I$. Prove that $A_1A_2 + B_1B_2 + C_1C_2 = P$, where P is the perimeter of $\triangle ABC$.

6. On each cell of a square 9×9 grid there is a trained beetle. Upon a whistle, each beetle moves to one of the neighbouring cells having a vertex but not an edge in common with the beetle's previous cell. The result is that some cells become empty and in some cells there are now several beetles. Find the minimal possible number of empty cells.

7. A function $f(x)$ is defined on the interval $[0,1]$, so that $f(0) = f(1) = 0$ and

$$f\left(\frac{a+b}{2}\right) \leq f(a) + f(b)\,.$$

for all a and b from $[0,1]$.

(a) Show that the equation $f(x) = 0$ has infinitely many solutions on $[0,1]$.

(b) Are there functions on $[0,1]$ which satisfy the above conditions but are not identically zero?

8. Prove that any prime number $2^{2^n}+1$ cannot be represented as a difference of two fifth powers of integers.

I want to apologize for overlooking some solutions from Michel Bataille, Rouen, France, which were lost in my filing system. Bataille's name should have been added to the list of solvers for the following problems:

- Yugoslav Qualification 2nd Round, Problem 1 [2005 : 374; 2006 : 507];
- 27ième Olympiade Belge, Problem 4 [2005 : 375 ; 2006 : 509] ;
- Bosnia and Herzegovina, National Olympiad Selection Test, Problems 2 and 4 [2005 : 436; 2007 : 22];
- 15th Irish Olympiad, Problems 1, 4, 5, 8, and 9 [2005 : 437–439; 2007 : 28, 30, 33].

Now we turn to solutions from our readers to problems of the 10th Grade Romanian Mathematical Olympiad given [2006 : 85–86].

1. Let $OABC$ be a tetrahedron such that $OA \perp OB \perp OC \perp OA$, let r be the radius of its inscribed sphere, and let H be the orthocentre of triangle ABC. Prove that $OH \le r(\sqrt{3}+1)$.

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Let Δ_A, Δ_B, Δ_C, and Δ_O be the area of triangles OBC, OAC, OAB, and ABC, respectively. Taking into account that triangles OBC, OAC, and OAB are projections of ABC in mutually orthogonal directions, we have

$$\Delta_O^2 \;=\; \Delta_A^2 + \Delta_B^2 + \Delta_C^2 \,.$$

Applying the AM–QM Inequality, we get

$$\Delta_O^2 \;=\; \Delta_A^2 + \Delta_B^2 + \Delta_C^2 \;\ge\; \tfrac{1}{3}(\Delta_A + \Delta_B + \Delta_C)^2 \,,$$

and hence,

$$\frac{\Delta_A + \Delta_B + \Delta_C}{\Delta_O} \;\le \sqrt{3}\,. \tag{1}$$

Since OH is perpendicular to triangle ABC, the volume of the tetrahedron $OABC$ is

$$\tfrac{1}{3}OH \cdot \Delta_O \;=\; \tfrac{1}{3}r(\Delta_A + \Delta_B + \Delta_C + \Delta_O)\,,$$

from which we get

$$OH \;=\; r\,\frac{\Delta_A + \Delta_B + \Delta_C + \Delta_O}{\Delta_O} \;=\; r\left(1 + \frac{\Delta_A + \Delta_B + \Delta_C}{\Delta_O}\right)\,.$$

Finally, using (1), we obtain $OH \le r(1+\sqrt{3})$.

2. The complex numbers $z_1, z_2, \ldots, z_5$ have the same non-zero modulus, and $\sum\limits_{i=1}^{5} z_i = \sum\limits_{i=1}^{5} z_i^2 = 0$. Prove that $z_1, z_2, \ldots, z_5$ are the complex coordinates of the vertices of a regular pentagon.

Solved by Michel Bataille, Rouen, France; and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. We give Bataille's solution.

Let r be the common modulus of $z_1, z_2, \ldots, z_5$ and $p = z_1 \cdot z_2 \cdots z_5$ be their product. Note that $|p| = r^5 \ne 0$.

The complex numbers $z_1, z_2, \ldots, z_5$ are the roots of the polynomial

$$\prod_{k=1}^{5}(z - z_k) \;=\; z^5 - \sigma_1 z^4 + \sigma_2 z^3 - \sigma_3 z^2 + \sigma_4 z - p\,,$$

where σ_k $(k = 1, 2, 3, 4)$ are the usual symmetric functions of the roots $z_1, z_2, \ldots, z_5$. By hypothesis, $\sigma_1 = 0$. Since $2\sigma_2 = \sigma_1^2 - \sum_{i=1}^{5} z_i^2$, we also have $\sigma_2 = 0$. Using $z_i \cdot \overline{z_i} = r^2$, we compute:

$$\sigma_3 = \sum z_i z_j z_k = \sum_{i<j} \frac{p}{z_i z_j} = \frac{p}{r^4} \sum_{i<j} \overline{z_i z_j} = \frac{p}{r^4} \cdot \overline{\sigma_2} = 0$$

$$\sigma_4 = \sum z_i z_j z_k z_l = \sum_{i=1}^{5} \frac{p}{z_i} = \frac{p}{r^2} \sum_{i=1}^{5} \overline{z_i} = \frac{p}{r^2} \cdot \overline{\sigma_1} = 0\,.$$

It follows that $z_1, z_2, \ldots, z_5$ are the fifth roots of the non-zero complex number p and, as such, are the complex coordinates of the vertices of a regular pentagon.

The next block of solutions are for problems of the 15th Korean Mathematical Olympiad appearing [2006 : 86–87].

1. The computers in a computer lab are connected by cables as follows: Each computer is directly connected to exactly three other computers via cables. There is at most one cable joining two computers and any pair of computers in the lab can exchange data. (Two computers A and B can exchange data if there exists a sequence of computers starting from A and ending at B in which two computers next to each other in the sequence are directly joined by a cable.)

Let k be the smallest number of computers in the lab whose removal results in leaving just one computer in the lab or a pair of computers not able to exchange data any more. Let ℓ be the smallest number of cables whose deletion results in the existence of two computers that cannot exchange data any more. Show that $k = \ell$.

Solution by Joan P. Hutchison, Macalester College, St. Paul, Minnesota, USA.

In the language of graph theory, this problem asserts that vertex-connectivity k equals edge-connectivity l in a connected 3–regular graph G. Three-regularity implies that $l \leq 3$. Further, $k \leq l$ because if a set of edges disconnects the graph, then there is a set of vertices chosen one from each edge that disconnects or leaves one vertex. Therefore, we have equality if $k = 3$ and need only consider $k \in \{1, 2\}$.

Suppose $k = 1$. Then there is a vertex v whose removal leaves two or three components. Because v has degree 3, there must be an edge incident with v whose removal disconnects the graph.

If $k = 2$, suppose the removal of vertices v and w disconnects the graph. If v is adjacent to w, then there are four edges joining $\{v, w\}$ to the rest of the graph. Since $k = 2$, $G \backslash \{v, w\}$ has two components, each with two edges to $\{v, w\}$. Either pair of edges disconnects the graph.

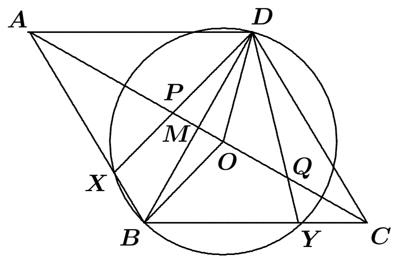
If v is not adjacent to w, then there are two or three components in $G\backslash\{v, w\}$. If three, there are six edges joining $\{v, w\}$ to the three components and they must be divided 2, 2, and 2. Each such pair of edges separates the graph. If there are two components A and B, then we may assume one edge at v connects to A, while the other two connect to B. Similarly, the edges incident with w are divided one and two. The removal of the two singleton edges disconnects the graph.

2. Let $ABCD$ be a rhombus with $\angle A < 90^\circ$. Let its two diagonals AC and BD meet at a point M. A point O on the line segment MC is selected such that $O \neq M$ and $OB < OC$. The circle centred at O passing through points B and D meets the line AB at point B and a point X (where $X = B$ when the line AB is tangent to the circle) and meets the line BC at point B and a point Y. Let the lines DX and DY meet the line segment AC at P and Q, respectively. Express the value of $\dfrac{OQ}{OP}$ in terms of t when $\dfrac{MA}{MO} = t$.

Solution by Mohammed Aassila, Strasbourg, France.

Since the quadrilateral $DXBY$ is cyclic, we have

$$\begin{aligned}\angle AXD &= \angle BYD \\ &= \angle BOA = \angle BOP\,;\end{aligned}$$



hence, quadrilateral $BOPX$ is cyclic. Consider the inversion I of pole O and power OB^2. Then the circle circumscribing $BOPX$ maps to the line AB. Thus, $I(P) = A$, which implies that $OP \cdot OA = OB^2$.

Similarly, we obtain $OQ \cdot OC = OB^2$. Therefore,

$$\frac{OQ}{OP} = \frac{OA}{OC} = \frac{MA + MO}{MA - MO} = \frac{\frac{MA}{MO} + 1}{\frac{MA}{MO} - 1} = \frac{t+1}{t-1}\,.$$

4. Suppose that the incircle of $\triangle ABC$ is tangent to the sides AB, BC, CA at points P, Q, R, respectively. Prove the following inequality:

$$\frac{BC}{PQ} + \frac{CA}{QR} + \frac{AB}{RP} \geq 6\,.$$

Solved by Mohammed Aassila, Strasbourg, France; and Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. We present the solution of Amengual Covas.

We use the standard notation a, b, c for the sides of triangle ABC, and s for the semiperimeter.

We have

$$\begin{aligned} PQ &= 2(s-b)\sin\frac{B}{2} = 2(s-b)\sqrt{\frac{(s-c)(s-a)}{ca}}\,, \\ QR &= 2(s-c)\sin\frac{C}{2} = 2(s-c)\sqrt{\frac{(s-a)(s-b)}{ab}}\,, \\ RP &= 2(s-a)\sin\frac{A}{2} = 2(s-a)\sqrt{\frac{(s-b)(s-c)}{bc}}\,. \end{aligned}$$

We apply the AM–GM Inequality and use the above expressions for PQ, QR, and RP to get

$$\frac{BC}{PQ}+\frac{CA}{QR}+\frac{AB}{RP} \ \ge\ 3\cdot\sqrt[3]{\frac{BC\cdot CA\cdot AB}{PQ\cdot QR\cdot RP}} \ =\ 3\cdot\sqrt[3]{\frac{(abc)^2}{8[(s-a)(s-b)(s-c)]^2}}\,.$$

Finally, we use the well-known inequality $abc \ge 8(s-a)(s-b)(s-c)$, which is equivalent to Euler's Inequality, to obtain

$$\frac{BC}{PQ}+\frac{CA}{QR}+\frac{AB}{RP} \ \ge\ 3\sqrt[3]{8} \ =\ 6\,.$$

Equality occurs only if $\triangle ABC$ is equilateral.

5. Answer the following where m is a positive integer.

(a) Prove that if $2^{m+1}+1$ divides $3^{2^m}+1$, then $2^{m+1}+1$ is a prime.

(b) Is the converse of (a) true?

Comment by Mohammed Aassila, Strasbourg, France.

Part (a) and its converse constitute Pepin's Test of primality for Fermat numbers (1877).

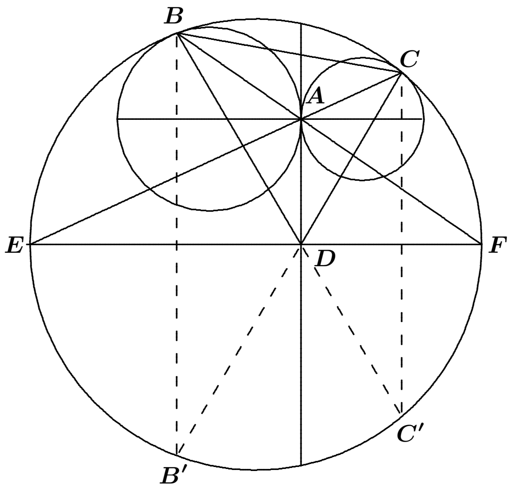
The next solutions are to problems of the X National Mathematical Olympiad of Turkey, given [2006 : 87–88].

2. Two circles are externally tangent to each other at a point A and internally tangent to a third circle Γ at points B and C. Let D be the mid-point of the secant of Γ which is tangent to the smaller circles at A. Show that A is the incentre of the triangle BCD if the centres of the circles are not collinear.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let E and F be the end-points of the diameter of Γ which is perpendicular to AD. Since the perpendicular bisector of a chord of a circle is a diameter of this circle, EF passes through D. The diameters of the smaller circles through A are perpendicular to AD and hence are parallel to EF.

Therefore points B, A, and F are collinear, and C, A, and E are collinear. (*Proof:* (1) Proposition 1 of the book of Lemmas of Archimedes states: "two circles touch at P and if TU, VW be parallel diameters in them, PUW is a right line". (2) Points A and B are corresponding points in the inversion centred at F with the power of the inversion equal to $FD \cdot FE$, and points A and C are corresponding points in the inversion centred at F with the power of the inversion equal to $ED \cdot EF$.)



Let B' and C' be the reflections of B and C, respectively, across EF. Then B' and C' lie on Γ. Since $BB'C'C$ is an isosceles trapezoid, BC' and $B'C$ intersect at D. Thus,

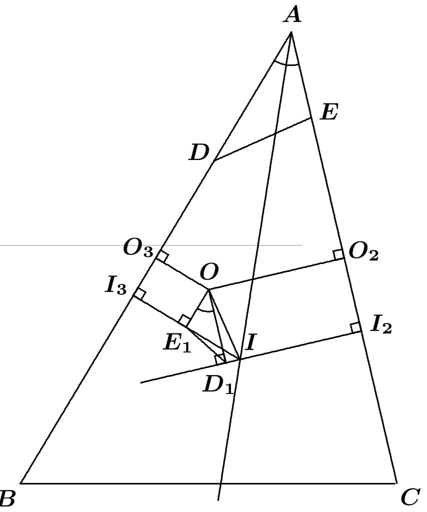
$$\begin{aligned} \angle CBA &= \angle CBF = \angle CEF = \angle FBC' = \angle ABD \\ \text{and} \quad \angle ACB &= \angle ECB = \angle EFB = \angle B'CE = \angle DCA\,, \end{aligned}$$

making A the incentre of triangle ABC.

5. In an acute triangle ABC with $|BC| < |AC| < |AB|$, the points D on side AB and E on side AC satisfy the condition $|BD| = |BC| = |CE|$. Show that the circumradius of the triangle ADE is equal to the distance between the incentre and the circumcentre of the triangle ABC.

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.

Let O be the circumcentre of $\triangle ABC$ and I its incentre. Let the projections of O and I onto AC be O_2 and I_2, respectively, and let the projections of O and I onto AB be O_3 and I_3, respectively. Let the projections of O onto II_2 and II_3 be D_1 and E_1, respectively.



We have

$$OD_1 = O_2I_2 = \tfrac{1}{2}(c-a) = \tfrac{1}{2}AD$$

and

$$OE_1 = O_3I_3 = \tfrac{1}{2}(b-a) = \tfrac{1}{2}AE\,.$$

Also, $\angle D_1OE_1 = \angle DAE$. Hence, $\triangle OD_1E_1$ is similar to $\triangle ADE$, and the scale factor of the similarity is $\frac{1}{2}$. Since

$$\angle OD_1I = \angle OE_1I = 90^\circ\,,$$

we see that OI is the diameter of the circumcircle of $\triangle OD_1E_1$. Then OI is the circumradius of $\triangle ADE$.

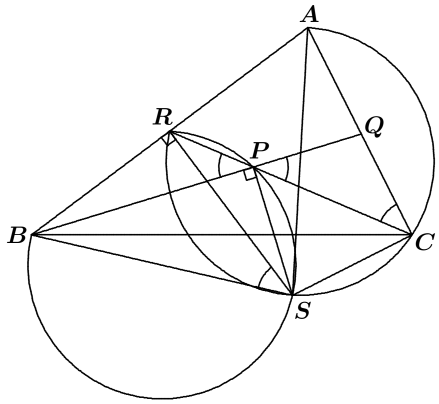
Remark. The points O, E_1, D_1, and I are concyclic if and only if $\angle OIE_1 = \angle OD_1E_1 = \angle ADE$. Since $IE_1 \perp AB$, it follows that $OI \perp DE$.

Next we turn to readers' solutions to problems given in the April 2007 number of the *Corner* and the Japan Mathematical Olympiad 2003 given at [2006 : 149–150].

1. A point P lies in a triangle ABC. The edge AC meets the line BP at Q, and AB meets CP at R. Suppose that $AR = RB = CP$ and $CQ = PQ$. Find $\angle BRC$.

Solved by Mohammed Aassila, Strasbourg, France; and Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. We give the solution of Amengual Covas.

Let S be the second point of intersection of the circumcircles of triangles BPR and RCA.



Since B, S, P, and R are concyclic, we have $\angle BSR = \angle BPR$; since A, R, S, and C are concyclic, $\angle RCA = \angle RSA$. Then

$$\angle BSR \;=\; \angle BPR \;=\; \angle QPC \;=\; \angle PCQ \;=\; \angle RCA \;=\; RSA\,.$$

Thus, SR bisects $\angle BSA$. Since $BR = RA$, the angle bisector theorem gives us $BS = SA$. Consequently, $\angle BRS = 90^\circ$. Then $\angle BPS = 90^\circ$ (because B, S, P, and R are concyclic).

We have

$$\begin{aligned} \angle CPS &= 90^\circ - \angle QPC \;=\; 90^\circ - \angle BSR \;=\; \angle RBS = \angle ABS \\ \text{and} \quad \angle SCP &= \angle SCR \;=\; \angle SAR \;=\; \angle SAB\,. \end{aligned}$$

Therefore, triangles ABS and CPS are similar. Since $\triangle ABS$ is isosceles with $BS = SA$, it follows that $\triangle CPS$ is isosceles with $PS = SC$. Also,

$$\frac{PS}{BS} \;=\; \frac{CP}{AB} \;=\; \frac{\frac{1}{2}AB}{AB} \;=\; \frac{1}{2}\,,$$

making $\angle SBP = 30^\circ$ in right triangle PBS. Therefore,

$$\begin{aligned} \angle BRC &= \angle BRS + \angle SRC \;=\; \angle BRS + \angle SRP \\ &= \angle BRS + \angle SBP \;=\; 90^\circ + 30^\circ \;=\; 120^\circ\,. \end{aligned}$$

3. Find the greatest real number k such that, for any positive a, b, c with $a^2 > bc$,

$$(a^2 - bc)^2 \;>\; k(b^2 - ca)(c^2 - ab)\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solution.

The greatest k is 4.

First suppose that $(a^2 - bc)^2 > k(b^2 - ca)(c^2 - ab)$ whenever a, b, $c > 0$ and $a^2 > bc$. Let $t \in (0,1)$. Since $1^2 > t \cdot t$, we have

$$(1 - t^2)^2 \;>\; k(t^2 - t)(t^2 - t)\,,$$

from which we deduce that

$$\left(\frac{1+t}{t}\right)^2 \;>\; k\,.$$

It follows that

$$k \;\le\; \lim_{t\to 1}\left(\frac{1+t}{t}\right)^2 \;=\; 4\,.$$

Now we will show that $(a^2 - bc)^2 > 4(b^2 - ca)(c^2 - ab)$ whenever a, b, $c > 0$ and $a^2 > bc$. Assume on the contrary that

$$(a^2 - bc)^2 \;\le\; 4(b^2 - ca)(c^2 - ab) \tag{1}$$

for some positive a, b, c such that $a^2 > bc$, and define

$$f(x) = (b^2 - ca)x^2 + (a^2 - bc)x + (c^2 - ab) .$$

From (1), either $f(x) \geq 0$ for all real x or $f(x) \leq 0$ for all real x. Actually, the former holds since $f(1) = a^2 + b^2 + c^2 - ab - bc - ca > 0$ (note that $a = b = c$ is excluded by $a^2 > bc$, and so $a^2 + b^2 + c^2 > ab + bc + ca$).

It follows that $b^2 - ca$ is positive. Now, write

$$f(x) = (bx - c)^2 - ag(x) - x(a^2 - bc) ,$$

where $g(x) = cx^2 - 2ax + b$. Since $a^2 - bc > 0$ and

$$g\left(\frac{c}{b}\right) = \frac{c(c^2 - ab) + b(b^2 - ac)}{b^2} > 0$$

(since $c^2 - ab$ has the same sign as $b^2 - ca$ by (1)), we have $f\left(\frac{c}{b}\right) < 0$, a contradiction. This completes the proof.

Next we look at solutions to problems of the Hungarian Mathematical Olympiad 2002–2003 First Round given at [2006 : 150].

1. A rectangular brick has volume $V = x$ cm^3, and surface area $S = y$ cm^2. Find the minimal volume for which $x = 10y$.

Solution by Houda Anoun, Bordeaux, France.

Let a, b, and c be the dimensions of the brick. Then $x = abc$ and $y = 2a^2 + 2b^2 + 2c^2$. By the AM–GM Inequality, we have

$$x^{2/3} = (a^2b^2c^2)^{1/3} \leq \frac{a^2 + b^2 + c^2}{3} = \frac{y}{6} .$$

When $x = 10y$, we get $x^{2/3} \leq \frac{x}{60}$; that is, $x \geq 60^3 = 216000$. Moreover, when $a = b = c = 60$ we have $x = 10y = 216000$. Hence, the minimal volume for which $x = 10y$ is 216000.

3. Let ABC be a triangle. We drop a perpendicular from A to the internal bisectors starting from B and C, their feet being A_1 and A_2. In the same way we define B_1, B_2 and C_1, C_2. Prove that

$$2(A_1A_2 + B_1B_2 + C_1C_2) = AB + BC + CA .$$

Solution by Michel Bataille, Rouen, France.

Let I be the incentre of ΔABC. Then,

$$\angle BAA_1 = \frac{\pi}{2} - \frac{B}{2} = \frac{A+C}{2} > \frac{A}{2} = \angle BAI\,,$$
$$\angle CAA_2 = \frac{\pi}{2} - \frac{C}{2} = \frac{A+B}{2} > \frac{A}{2} = \angle CAI\,.$$

Thus, A_1 and A_2 are on opposite sides of the bisector AI. Moreover,

$$\angle IAA_1 = \frac{\pi}{2} - \angle AIA_1 = \frac{\pi}{2} - \left(\frac{A}{2} + \frac{B}{2}\right) = \frac{C}{2}$$

(since $\angle AIA_1 = \pi - \angle AIB$) and similarly, $\angle IAA_2 = \frac{B}{2}$. It follows that

$$IA_1 = AI\sin\left(\frac{C}{2}\right) \quad \text{and} \quad IA_2 = AI\sin\left(\frac{B}{2}\right)\,. \qquad (1)$$

With the familiar notations $a = BC$, $b = CA$, and $c = AB$, we also have

$$AA_1 = c\sin\left(\frac{B}{2}\right) \quad \text{and} \quad AA_2 = b\sin\left(\frac{C}{2}\right)\,. \qquad (2)$$

Now, observe that A, A_2, I, and A_1 are (in this order) on the circle with diameter AI. Ptolemy's Theorem gives $A_1A_2 \cdot AI = IA_2 \cdot AA_1 + IA_1 \cdot AA_2$, from which, using (1) and (2), we get

$$A_1A_2 = c\sin^2\left(\frac{B}{2}\right) + b\sin^2\left(\frac{C}{2}\right)\,.$$

There are analogous results for B_1B_2 and C_1C_2. Now we have

$$\begin{aligned}
&2(A_1A_2 + B_1B_2 + C_1C_2)\\
&\quad= 2(a+b)\sin^2\left(\frac{C}{2}\right) + 2(b+c)\sin^2\left(\frac{A}{2}\right) + 2(c+a)\sin^2\left(\frac{B}{2}\right)\\
&\quad= (a+b)(1-\cos C) + (b+c)(1-\cos A) + (c+a)(1\cos B)\\
&\quad= 2(a+b+c) - \big[(c\cos B + b\cos C)\\
&\qquad\qquad + (a\cos C + c\cos A) + (a\cos B + b\cos A)\big]\\
&\quad= 2(a+b+c) - (a+b+c) = a+b+c = AB + BC + CA\,.
\end{aligned}$$

Next we look at solutions for the Hungarian Mathematical Olympiad 2002–2003, Final Round given at [2006 : 151].

2. We colour the vertices of a 2003-gon with red, blue, and green such that neighbours cannot have the same colour. In how many ways can we accomplish this?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Consider the graph whose vertices are the vertices of the 2003–gon and whose edges are the edges of the 2003–gon. This graph is the cycle $\mathcal{C}_{2003}$, and the problem is to determine the number of its proper colourings.

It is a classical exercise ([1], [2]) to prove that, more generally, the number of proper colourings of $\mathcal{C}_n$ with q colours is

$$P_{\mathcal{C}_n}(q) \;=\; (q-1)^n + (-1)^n(q-1)$$

(called the chromatic polynomial of C_n). In particular, $P_{\mathcal{C}_{2003}}(3) = 2^{2003} - 2$.

References

[1] L. Lovász, *Combinatorial problems and exercises*, North-Holland, exercise 9–39.

[2] I. Tomescu, *Problems in combinatorics and graph theory*, Wiley, exercise 10–16.

3. Let t be a fixed positive integer. Let $f_t(n)$ denote the number of integers k such that $1 \le k \le n$ and $\binom{k}{t}$ is odd. (If $1 \le k < t$, then $\binom{k}{t} = 0$.) Prove that if n is a sufficiently great power of 2, then $\frac{f_t(n)}{n} = \frac{1}{2^r}$, where r is an integer determined by t and independent of n.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let $n = 2^p$ with $2^p > 2t$. Let $t = \sum_{i=0}^{p} t_i 2^i$, where $t_i \in \{0, 1\}$ for each i. Note that this is the binary expansion of t except that t_i can be zero for some $i = p, p-1, \ldots$. Let r be the number of i such that $t_i = 1$. Then r is the number of 1s in the binary expansion of t; thus, r is clearly independent of n.

For $1 \le k \le n$, let $k = \sum_{i=0}^{p} k_i 2^i$, where $k_i \in \{0, 1\}$ for each i. Recall Lucas' Theorem (see [1]), which states that

$$\binom{k}{t} \;\equiv\; \prod_{i=0}^{p} \binom{k_i}{t_i} \pmod{2} \;.$$

Thus, $\binom{k}{t}$ is odd if and only if $k_i \ge t_i$ for each i. Therefore, $\binom{k}{t}$ is odd if and only if $k_i = 1$ when $t_i = 1$, and k_i can be either 0 or 1 otherwise. It follows from the definition of r that there are exactly 2^{p-r} integers k such that $1 \le k \le n$ and $\binom{k}{t}$ is odd (note that $r \ge 1$ since $t > 0$, which ensures that at least one of the k_is is non-zero, so that $k \ge 1$). Then

$$\frac{f_t(n)}{n} \;=\; \frac{2^{p-r}}{2^p} \;=\; \frac{1}{2^r} \,.$$

References

[1] T. Andreescu, R. Gelca, *Mathematical Olympiad Challenges*, Birkhäuser, p. 84.

To complete this number of the *Corner*, we look at solutions to the 2002 Kürschák Competition given at [2006 : 151].

2. The Fibonacci sequence is defined by the following recursion: $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$. Suppose that the positive integers a and b satisfy:

$$\min\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\} \le \frac{a}{b} \le \max\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\}.$$

Prove that $b \ge f_{n+1}$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The statement is false, as we can see by choosing $n = 2$ and $a = b = 1$. The correct statement is the one with strict inequalities:

$$\min\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\} < \frac{a}{b} < \max\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\}.$$

Lemma 1. Let x, y, z, t, a, and b be positive integers such that $yz - xt = 1$ and $\frac{x}{y} < \frac{a}{b} < \frac{z}{t}$. Then $b \ge y + t$.

Proof: Since $xb < ay$ and all the numbers are integers, we deduce that $xb \le ay - 1$. Similarly, $ta \le bz - 1$. Therefore,

$$txb \le t(ay - 1) = tay - t \le (bz - 1)y - t = bzy - (y + t),$$

which gives $y + t \le b(yz - tx) = b$. ∎

Lemma 2. For each $n > 1$, we have $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$.

Proof: The proof is by induction on n.

We have $f_3 f_1 - f_2^2 = 2 \cdot 1 - 1^2 = (-1)^2$. Hence, the result is true for $n = 2$.

Assume that the result holds for some given $n > 1$. Then

$$\begin{aligned} f_{n+2}f_n - f_{n+1}^2 &= (f_{n+1} + f_n)f_n - f_{n+1}(f_n + f_{n-1}) \\ &= -(f_{n+1}f_{n-1} - f_n^2) = -(-1)^n = (-1)^{n+1}, \end{aligned}$$

which completes the induction. ∎

Now assume that

$$\min\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\} < \frac{a}{b} < \max\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\}.$$

Case 1. n is even.

From Lemma 2, we have $f_{n+1}f_{n-1} - f_n^2 = 1 \geq 0$, and therefore, $\frac{f_{n+1}}{f_n} \geq \frac{f_n}{f_{n-1}}$. Thus,

$$\frac{f_n}{f_{n-1}} < \frac{a}{b} < \frac{f_{n+1}}{f_n}.$$

Then, from Lemma 1, we have $b \geq f_n + f_{n-1} = f_{n+1}$, as desired.

Case 2. n is odd.

Arguing as in Case 1, we obtain

$$\frac{f_{n+1}}{f_n} < \frac{a}{b} < \frac{f_n}{f_{n-1}},$$

and the desired conclusion follows once again from Lemma 1.

3. Prove that one can distribute all the sides and diagonals of a convex 3^n–gon into groups of three segments such that in each group the three segments form a triangle.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We shall prove that the result holds for any convex $3k$–gon, where k is an odd integer.

Let $A_1, \ldots, A_k, B_1, \ldots, B_k, C_1, \ldots, C_k$ be the vertices of the $3k$–gon, in any order. Since k is odd, it follows that 2 is invertible modulo k. Let $\frac{1}{2}$ be its inverse. All subscripts are considered modulo k.

Note that for $1 \leq i < j \leq k$, we have $\frac{1}{2}(i+j) \not\equiv i \pmod{k}$ and $\frac{1}{2}(i+j) \not\equiv j \pmod{k}$. Moreover, for a given i and $m \not\equiv i \pmod{k}$, there exists a unique $j \not\equiv i \pmod{k}$ such that $m \equiv \frac{1}{2}(i+j) \pmod{k}$.

For $1 \leq i \leq k$, we form the triangles $A_iB_iC_i$; for $1 \leq i < j \leq k$, we form the triangles $A_iA_jB_{\frac{1}{2}(i+j)}$, $B_iB_jC_{\frac{1}{2}(i+j)}$, and $C_iC_jA_{\frac{1}{2}(i+j)}$. Now a straightforward verification shows that these triangles use each side and diagonal of the $3k$–gon exactly once, as desired.

The backlog of solutions is now cleared. Please send me your nice solutions and generalizations soon for use in the *Corner*.