

## Problem of the Month

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This month, we have a couple of problems demonstrating that knowing too much algebra may be dangerous!

**Problem #1** (1974 Gauss Contest). A car is driven up a 1 km long hill at 30 km/h, and continues down the other side, which is also 1 km in length. The speed the car must be driven on the down slope, in km/h, in order to average 60 km/h for the whole trip is

- (A) 30      (B) 90      (C) 60      (D) 120      (E) none of these

It is tempting to answer 90 km/h, since the average of 30 and 90 is 60, but this somehow seems too easy. (To boot, you probably have that nagging voice in the back of your head reminding you of something your Grade 5 mathematics teacher told you about this sort of problem. . . .)

The most important thing to remember in solving this problem is that speed equals distance divided by time, which is the same as distance equals speed multiplied by time, or time equals distance divided by speed.

*Solution #1:* To drive up the 1 km hill at 30 km/h takes  $\frac{1}{30}$  hour, or 2 minutes. To average 60 km/h over the whole 2-km trip, the total driving time must be  $\frac{2}{60}$  hour, or 2 minutes. Then the downhill part of the trip must take  $2 - 2 = 0$  minutes. Since this is not possible, the answer must be (E).

Now, that was a surprising answer!

If we know some algebra (and try to use it), the solution becomes a bit more complicated.

*Solution #2:* Suppose the car is driven down the hill at  $v$  km/h. To find the average speed, we find the total distance driven (2 km in this case) and divide by the total time. The time for the uphill portion is  $\frac{1}{30}$  hour as in Solution #1 above. The time for the downhill portion is  $1/v$  hour. Therefore, the total time is  $\frac{1}{30} + \frac{1}{v}$  hour. Hence, the average speed, in km/h, is

$$\frac{2}{\frac{1}{30} + \frac{1}{v}} = 60,$$

which simplifies to

$$2 = 60 \left( \frac{1}{30} + \frac{1}{v} \right) = 2 + \frac{60}{v}.$$

Thus,  $60/v = 0$ , which is impossible.

Let's try a couple of variations of the problem. First, try the problem with 40 km/h instead of 30 km/h. Try it by both of the methods used above. Did you get 120 km/h? (Going uphill for 1 km at 40 km/h should take  $\frac{1}{40}$  of an hour, which is  $1\frac{1}{2}$  minutes, leaving half a minute out of the total of 2 minutes to drive the remaining 1 km downhill.)

What happens if we replace the 30 km/h with 20 km/h? If we proceed mathematically (without thinking), we find that the car must be driven down the hill at  $-60$  km/h. One wonders what that really means! This shows that we always need to think about what we are doing.

Suppose the car is driven uphill at  $u$  km/h and then downhill at  $v$  km/h. What are the possible values of  $u$  that allow us to solve the problem (that is, to get a positive value for  $v$ )?

We model Solution #2 from above. The time to drive uphill is  $1/u$  hours and to drive downhill is  $1/v$  hours. Therefore, the total driving time is  $(1/u) + (1/v)$  hours, and the average speed is

$$\frac{2}{\frac{1}{u} + \frac{1}{v}} = 60.$$

Now we solve for  $v$  in terms of  $u$ :

$$\begin{aligned} 2 &= 60 \left( \frac{1}{u} + \frac{1}{v} \right), \\ \frac{1}{30} - \frac{1}{u} &= \frac{1}{v}, \\ v &= \frac{1}{\frac{1}{30} - \frac{1}{u}} = \frac{30u}{u - 30}. \end{aligned}$$

We know that  $u$  is a positive real number. For  $v$  to be a positive real number, we need  $u - 30 > 0$ , or  $u > 30$ . Thus, an uphill speed of more than 30 km/h allows us to find a downhill speed that gives an average speed of 60 km/h.

For what *integer* values of  $u$  can we find a positive integer value of  $v$  that gives an average speed of 60 km/h? We know already that

$$v = \frac{30u}{u - 30} = \frac{30u - 900 + 900}{u - 30} = 30 + \frac{900}{u - 30}.$$

For  $v$  to be an integer, we need  $\frac{900}{u - 30}$  to be an integer, and for this we need  $u - 30$  to be a divisor of 900. You can list out the divisors of 900 and the corresponding values of  $u$ .

So, a problem that starts out being quite simple has lots of interesting ideas that can be gleaned from it. The most important thing to remember here is the very first simple solution. Thinking about this type of problem in a clever way will often get you the answer more easily than using a formal algebraic approach.

Let's try to apply this way of thinking to another problem.

**Problem #2.** Jeff is on a railway bridge joining  $A$  to  $B$ , and is  $\frac{3}{8}$  of the way across from  $A$ . He hears a train approaching  $A$ ; it is travelling at 80 km/h. If he runs towards  $A$ , he will meet the train at  $A$ . If he runs towards  $B$ , the train will overtake him at  $B$ . How fast can he run?

*Solution #1:* Let's try this algebraically first. Suppose the bridge has a length  $R$ , the train is a distance  $d$  from point  $A$ , and Jeff's running speed is  $v$ . The amount of time it would take Jeff to run to  $A$  is  $\frac{3}{8}R/v$  and to run to  $B$  is  $\frac{5}{8}R/v$ . (Each of these quantities is distance divided by speed.) The amount of time it would take the train to get to  $A$  is  $d/80$  and to get to  $B$  is  $(d + R)/80$ . Since Jeff and the train would arrive at  $A$  or  $B$  at the same time,

$$\frac{\frac{3}{8}R}{v} = \frac{d}{80} \quad \text{and} \quad \frac{\frac{5}{8}R}{v} = \frac{d + R}{80}.$$

Now we have two equations with three unknowns and want to solve for  $v$ . Try fiddling with these before going on!

Any luck? We could solve this by substituting one equation into the other, but here's a more clever way. By subtracting the first equation from the second one, we get

$$\frac{\frac{1}{4}R}{v} = \frac{R}{80},$$

which yields  $v = 20$  km/h. Hence, Jeff runs at 20 km/h.

But perhaps we can find a better method. How about this?

*Solution #2:* In the amount of time that Jeff runs  $\frac{3}{8}$  of the way across the bridge, the train gets to  $A$ . Suppose that Jeff runs  $\frac{3}{8}$  of the way across the bridge towards  $B$ , not  $A$ . Once Jeff has run this distance, he is  $\frac{3}{8} + \frac{3}{8} = \frac{3}{4}$  of the way from  $A$  to  $B$ , and the train is at  $A$ . But we also know that Jeff and the train get to  $B$  at the same time with Jeff running in this direction. Therefore, the train will travel the length of the entire bridge while Jeff runs the remaining  $\frac{1}{4}$  of the length of the bridge. Thus, the train's speed is 4 times Jeff's speed; that is, Jeff's speed is  $\frac{80}{4} = 20$  km/h.

Isn't that nice? To finish off, here is a problem of the same type for you to try:

A train passes completely through a tunnel in 10 minutes. A second train, twice as long, passes through the tunnel in 11 minutes. If both trains are travelling at the same speed, 72 km/h, determine the length of the tunnel and the lengths of the trains.