THE OLYMPIAD CORNER

No. 260

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We begin this number with the problems of the Second Round and Final Round of the Hungarian National Olympiad Grades 11–12 for 2003–2004. Thanks again go to Christopher Small, Canadian Team leader to the IMO in Athens, Greece in 2004, for collecting them for our use.

HUNGARIAN NATIONAL OLYMPIAD 2003–2004
Grades 11–12, Round 2

1. Let n be an integer, $n > 1$. Define

$$\begin{aligned} A &= \frac{\sqrt{n+1}}{n} + \frac{\sqrt{n+4}}{n+3} + \frac{\sqrt{n+7}}{n+6} + \frac{\sqrt{n+10}}{n+9} + \frac{\sqrt{n+13}}{n+12} \\ \text{and} \quad B &= \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n+2}} + \frac{1}{\sqrt{n+5}} + \frac{1}{\sqrt{n+8}} + \frac{1}{\sqrt{n+11}}\,. \end{aligned}$$

Determine which of the following relations holds (depending on n): $A > B$, $A = B$, or $A < B$.

2. Let a, b, and c denote the sides of a triangle opposite the angles A, B, and C, respectively. Let r be the inradius and R the circumradius of the triangle. If $\angle A \geq 90^\circ$, prove that

$$\frac{r}{R} \leq \frac{a \sin A}{a+b+c}\,.$$

3. Prove that the equation $x^3 + 2px^2 + 2p^2x + p = 0$ cannot have three distinct real roots, for any real number p.

4. Let $ABCD$ be a cyclic quadrilateral with $AB = 2AD$ and $BC = 2CD$. Let $d = AC$ and $\alpha = \angle BAD$ be given. Express the area of $ABCD$ in terms of d and α.

Grades 11–12, Final Round

1. Let ABC be an acute triangle, and let P be a point on side AB. Draw lines through P parallel to AC and BC, and let them cut BC and AC at X and Y, respectively. Construct (with straightedge and compass) the point P which gives the shortest length XY. Prove that the shortest XY is perpendicular to the median of ABC through C.

2. Let a, b, and c be distinct positive integers which are the side lengths of a triangle. There is a line which cuts both the area and the perimeter of the triangle into two equal parts. This line cuts the longest side of the triangle into two parts with ratio $2:1$. Determine a, b, and c for which abc is minimal.

3. Let $H = \{1, 2, 3, \ldots, 2004\}$. We denote by D the number of subsets of H such that the sum of the elements of the subset has a remainder of 7 when divided by 32. We denote by S the number of subsets of H such that the sum of the elements of the subset has a remainder of 14 when divided by 16. Prove that $S = 2D$.

Next we give the problems of the First Round (Specialized Mathematics Classes), Grades 11–12 of the Hungarian National Olympiad. Thanks again go to Christopher Small, Canadian Team Leader to the IMO in Athens, Greece in 2004, for collecting the set.

HUNGARIAN NATIONAL OLYMPIAD 2003–2004
(Specialized Mathematics Classes) Grades 11–12
First Round

1. Let n be a positive integer, and let a and b be positive real numbers. Prove that

$$\log(a^n) + \binom{n}{1}\log(a^{n-1}b) + \binom{n}{2}\log(a^{n-2}b^2) + \cdots + \log(b^n) = \log\left((ab)^{n2^{n-1}}\right).$$

2. Let H be a finite set of positive integers none of which has a prime factor greater than 3. Show that the sum of the reciprocals of the elements of H is smaller than 3.

3. Consider the three disjoint arcs of a circle determined by three points on the circle. For each of these arcs, draw a circle at the mid-point of the arc and passing through the end-points of the arc. Prove that the three circles have a common point.

4. A palace which has a square shape is divided into 2003×2003 square rooms of the same size which form a square grid. There might be a door between two rooms if they have a common side. The main gate leads to the room at the northwest corner. Someone has entered the palace, walked around for a while and upon returning to the room at the northwest corner for the first time, immediately left the palace. It turned out that this person visited each of the other rooms 100 times, except the room at the southeast corner. How many times did this person visit the room at the southeast corner?

5. Let a_0, a_1, $\dots$, a_n, a_{n+1} be real numbers such that $a_0 = a_{n+1} = 0$. Prove that there is a number k $(0 \le k \le n)$ such that

(a) $a_{k+1} + \cdots + a_{k+i} \ge 0$ for every $i = 1, \dots, n-k+1$, and

(b) $a_j + \cdots + a_k \le 0$ for every $j = 0, \dots, k$.

To complete the problems section we give the Final Round of the Finnish High School Mathematics Contest. My thanks go to Matti Lehtinen, Helsinki, Finland; and to Christopher Small, Canadian Team Leader to the IMO in Athens, Greece in 2004, for collecting them for our use.

FINNISH HIGH SCHOOL MATH CONTEST 2004

Final Round

February 6, 2004 – Time allowed: 3 hours

1. The equations

$$x^2 + 2ax + b^2 = 0 \qquad \text{and} \qquad x^2 + 2bx + c^2 = 0$$

both have two different real roots. Determine the number of real roots of the equation

$$x^2 + 2cx + a^2 = 0\,.$$

2. Let a, b, and c be positive integers such that

$$\frac{a\sqrt{3} + b}{b\sqrt{3} + c}$$

is a rational number. Show that

$$\frac{a^2 + b^2 + c^2}{a + b + c}$$

is an integer.

3. Two circles with radii r and R are externally tangent at a point P. Determine the length of the segment cut from the common tangent through P by the other common tangents.

4. The numbers $2005! + 2$, $2005! + 3$, $\dots$, $2005! + 2005$ form a sequence of 2004 consecutive integers, none of which is a prime number. Does there exist a sequence of 2004 consecutive integers containing exactly 12 prime numbers?

5. Finland is going to change its monetary system again and replace the Euro by the Finnish Mark. The Mark is divided into 100 pennies. There shall be coins of three denominations only, and the number of coins a person has to carry in order to be able to pay for any purchase less than one Mark should be minimal. Determine the coin denominations.

Now we turn to readers' solutions to problems given in the December 2005 number of the *Corner*, for the 38th Mongolian Mathematical Olympiad, Final Round, appearing [2006 : 505].

1. Let n and k be natural numbers. Find the least possible value for the cardinality of a set A that satisfies the following condition: There exist subsets $A_1, \ldots, A_n$ of A such that any union of k of the A_i is equal to A, but any union of $k-1$ of them is not equal to A.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

The answer is $\binom{n}{k-1}$.

This problem is equivalent to problem #6 of the final round of the 8th Korean Mathematical Olympiad. A proof appears in [2000 : 11].

2. For a natural number p, one can move between two integer points in a plane when the distance between the points is p. Find all primes p for which the point $(2002, 38)$ can be reached from the point $(0, 0)$ using permitted moves.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The desired primes are those not of the form $4k+3$.

Let p be a prime.

Case 1. $p \equiv 3 \pmod 4$.

It is well known that p is not the sum of two squares. Therefore, moves can only be done in the directions of the coordinate axes. In that case, both coordinates of any point which can be reached must be multiples of p. But $2002 = 2 \times 7 \times 11 \times 13$ and $38 = 2 \times 19$ have no common prime divisor of the form $4k+3$. Thus, it is impossible to reach $(2002, 38)$ for such a p.

Case 2. $p = 2$.

One can reach $(2002, 38)$ by 1001 moves of the form $(x, y) \to (x+2, y)$, and 19 moves of the form $(x, y) \to (x, y+2)$.

Case 3. $p \equiv 1 \pmod 4$.

It is well known that there exist two positive integers a and b such that $p = a^2 + b^2$. Moreover, since p is prime, we must have a and b coprime, and one of them, say a, is even.

Let us consider moves of the following four types:

1. $(x, y) \to (x+a, y+b)$,
2. $(x, y) \to (x+a, y-b)$,
3. $(x, y) \to (x+b, y+a)$,
4. $(x, y) \to (x+b, y-a)$.

We will prove that we can reach $(2002, 38)$ from $(0, 0)$ using only these four moves and their inverses (where the inverse of $(x, y) \to (x+a, y+b)$ is $(x, y) \to (x-a, y-b)$).

Let x_i denote the number of moves of type i (where x_i may be negative, meaning that we use the inverse of the move of type i $|x_i|$ times). We want to prove that there exist integers x_1, x_2, x_3, x_4 such that

$$2002 = a(x_1+x_2)+b(x_3+x_4) \quad \text{and} \quad 38 = a(x_3-x_4)+b(x_1-x_2)\,.$$

But, since a and b are coprime, there exist integers α, β, γ, δ such that $2002 = a\alpha + b\beta$ and $38 = a\gamma + b\delta$. Note that β and δ are even. Then, for all integers m and n, we have

$$2002 = a(\alpha+bm)+b(\beta-am) \quad \text{and} \quad 38 = a(\gamma+nb)+b(\delta-na)\,.$$

Hence, we want to find integer solutions to the system

$$\begin{aligned} x_1+x_2 &= \alpha+bm\,,\\ x_1-x_2 &= \delta-na\,,\\ x_3+x_4 &= \beta-am\,,\\ x_3-x_4 &= \gamma+nb\,. \end{aligned}$$

But the only condition which has to be satisfied to solve this system in the integers is that $\alpha+bm$ and $\gamma+nb$ are even. Since b is odd, this can be done by suitable choices for m and n, and we are done.

4. Given are 131 distinct natural numbers, each with prime divisors not exceeding 42. Prove that four of them can be chosen whose product is a perfect square.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let x_1, $\ldots$, x_{131} be the given numbers. Direct checking shows that there are exactly 13 primes not exceeding 42. Denote them by p_1, $\ldots$, p_{13}.

Consider the $\frac{131\times 130}{2} = 8515$ products x_ix_j (with repetition if any) of any two of the given numbers. Consider each of these products, say P, as a 13–tuple $(a_1,\ldots,a_{13})$, where a_i is the exponent, reduced modulo 2, of p_i in the prime decomposition of P (thus, $a_i \in \{0,\,1\}$). Since $8515 > 8192 = 2^{13}$, two of these products, say x_ix_j and x_mx_n, must be associated with the same 13–tuple. If $\{i,\,j\}\cap\{m,\,n\} = \emptyset$, this ensures that $x_ix_jx_mx_n$ is a perfect square, and we are done.

Otherwise, without loss of generality, we may assume that $j = n$, which means that x_ix_m is associated with $(0,0,\ldots,0)$. In that case, omit x_i and x_m from the given numbers and repeat the above procedure. Since $\frac{129\times 128}{2} = 8256 > 2^{13}$, we may find again two products, say x_rx_s and x_tx_u which are associated with the same 13–tuple. If $\{r,\,s\}\cap\{t,\,u\} = \emptyset$, this ensures that $x_rx_sx_tx_u$ is a perfect square, and we are done. Otherwise, without loss of generality, we may assume that $s = u$, which means that x_rx_t is associated with $(0,0,\ldots,0)$. Then $x_ix_mx_rx_t$ is the desired square.

5. Let a_0, a_1, $\ldots$ be an infinite sequence of positive real numbers. Show that $1+a_n > \sqrt[n]{2}\,a_{n-1}$ for infinitely many positive integers n.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Suppose, for the purpose of contradiction, that there exists $n_0 \geq 0$ such that $1 + a_n \leq \sqrt[n]{2}a_{n-1}$ for all $n > n_0$. We have $\sqrt[n]{2} \leq 1 + \frac{1}{n}$, from Bernoulli's Inequality. Hence, for all $n > n_0$,

$$a_n \leq \frac{n+1}{n}a_{n-1} - 1\,. \qquad (1)$$

We prove by induction on $p \geq 1$ that

$$a_{n_0+p} \leq (n_0+p+1)\left(\frac{a_{n_0}}{n_0+1} - \sum_{k=n_0+2}^{n_0+p+1}\frac{1}{k}\right). \qquad (2)$$

This is true for $p = 1$, because in this case it is just (1) with $n = n_0 + 1$. Now let us assume it holds for some given $p \geq 1$. Using (1) with $n = n_0 + p + 1$ and then applying the induction hypothesis, we get

$$\begin{aligned} a_{n_0+p+1} &\leq \frac{n_0+p+2}{n_0+p+1}a_{n_0+p} - 1 \\ &\leq (n_0+p+2)\left(\frac{a_{n_0}}{n_0+1} - \sum_{k=n_0+2}^{n_0+p+1}\frac{1}{k}\right) - 1 \\ &= (n_0+p+2)\left(\frac{a_{n_0}}{n_0+1} - \sum_{k=n_0+2}^{n_0+p+2}\frac{1}{k}\right), \end{aligned}$$

which ends the induction.

It is well known that $\sum \frac{1}{k}$ diverges to $+\infty$. For sufficiently large p,

$$\sum_{k=n_0+2}^{n_0+p+1}\frac{1}{k} > \frac{a_{n_0}}{n_0+1}\,.$$

For such a p, the inequality (2) forces a_{n_0+p} to be negative, a contradiction.

Next we look at a solution from a reader to problem 1 of the 19th Balkan Mathematical Olympiad, which appeared [2005 : 506].

1. Let A_1, A_2, …, A_n $(n \geq 4)$ be points in the plane such that no three of them are collinear. Some pairs of distinct points among A_1, A_2, …, A_n are connected by line segments in such a way that each point is connected to at least three others. Prove that there exists $k > 1$ and distinct points X_1, X_2, …, $X_{2k} \in \{A_1, A_2, \ldots, A_n\}$ such that for each $1 \leq i \leq 2k-1$, X_i is connected to X_{i+1} and X_{2k} is connected to X_1.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

An equivalent formulation is the following: A finite simple graph for which each vertex has degree at least 3 contains an even cycle.

Consider the longest path (using pairwise distinct vertices) in the graph, say X_1, X_2, $\ldots$, X_p. According to the maximality of the path, each vertex adjacent to X_1 must belong in the path. Since X_1 has degree at least 3, X_1 is adjacent to X_r and X_s, where $2 < r < s$.

Now, among the three integers 2, r, and s, at least two have the same parity. Then, X_1 is adjacent to X_a and to X_b, where $2 \leq a < b \leq p$ and $a \equiv b \pmod 2$. Thus, X_1–X_a–X_{a+1}–$\ldots$–X_b–X_1 is the desired even cycle.

Now we move to solutions from readers to problems of the Bulgarian Mathematical Olympiad, Final Round, 2003, given [2005 : 506–507].

3. Given the sequence $\{y_n\}_{n=1}^{\infty}$ defined by $y_1 = y_2 = 1$ and

$$y_{n+2} \;=\; (4k-5)y_{n+1} - y_n + 4 - 2k\,, \qquad n \;\geq\; 1\,,$$

find all integers k such that every term of the sequence is a perfect square.

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Pierre Bornsztein, Maisons-Laffitte, France. We give the write-up of Bornsztein.

The desired values are $k = 1$ and $k = 3$.

Assume that k is an integer such that $\{y_n\}_{n=1}^{\infty}$ contains only perfect squares. Then $y_3 = 2k - 2$ is an even square, say $4a^2$. Thus, $k = 2a^2 + 1$ for some non-negative integer a.

Moreover, the recurrence relation yields $y_4 = 8k^2 - 20k + 13$ and $y_5 = 32k^3 - 120k^2 + 148k - 59$. Thus, $y_5 = 256a^6 - 96a^4 + 8a^2 + 1$.

But, if $a \geq 2$, we have

$$256a^6 - 96a^4 + 8a^2 + 1 \;<\; 256a^6 - 96a^4 + 9a^2 \;=\; (16a^2 - 3a)^2\,,$$

while, since $a(32a^2 - a - 6) > 0$,

$$\begin{aligned} 256a^6 - 96a^4 + 8a^2 + 1 \;&>\; 256a^6 - 96a^4 - 32a^3 + 9a^2 + 6a + 1 \\ &=\; (16a^2 - 3a - 1)^2\,. \end{aligned}$$

Thus, $(16a^3 - 3a - 1)^2 < y_5 < (16a^3 - 3a)^2$, which contradicts the fact that y_5 is square.

Therefore, $a \in \{0, 1\}$, which leads to $k = 1$ or $k = 3$.

Conversely, consider the cases $k = 1$ and $k = 3$.

Case 1. $k = 1$.

One can verify by induction that the sequence $\{y_n\}_{n=1}^{\infty}$ is 1, 1, 0, 1, 1, 0, 1, 1, 0, $\ldots$ which is periodic with period 3. Since it contains only squares, $k = 1$ is a solution of the problem.

Case 2. $k = 3$.

Then $y_{n+2} = 7y_{n+1} - y_n - 2$ for $n \geq 1$. We will prove that, for all $n \geq 1$, we have $y_n = x_n^2$ where

$$x_1 \;=\; x_2 \;=\; 1 \qquad \text{and} \qquad x_{n+2} \;=\; 3x_{n+1} - x_n\,. \tag{1}$$

First, we prove that for the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (1), we have

$$x_{n+1}^2 + x_n^2 + 1 \;=\; 3x_{n+1}x_n \qquad \text{for all } n \geq 1. \tag{2}$$

This is clearly true for $n = 1$. Assume that it holds for some given $n \geq 1$. Then

$$\begin{aligned} x_{n+2}^2 + x_{n+1}^2 + 1 &= x_{n+2}^2 + 3x_{n+1}x_n - x_n^2 \quad \text{(induction hypothesis)} \\ &= x_{n+2}^2 + x_n(3x_{n+1} - x_n) \\ &= x_{n+2}^2 + x_nx_{n+2} \quad \text{from (1)} \\ &= x_{n+2}(x_{n+2} + x_n) \;=\; 3x_{n+2}x_{n+1} \quad \text{from (1).} \end{aligned}$$

This proves the relation for $n + 1$, and ends the induction.

It follows that, for all $n \geq 1$, we have:

$$\begin{aligned} x_{n+2}^2 &= (3x_{n+1} - x_n)^2 \;=\; 7x_{n+1}^2 + (2x_{n+1}^2 + x_n^2 - 6x_{n+1}x_n) \\ &= 7x_{n+1}^2 - x_n^2 - 2 \quad \text{from (2).} \end{aligned}$$

Hence, $\{x_n^2\}_{n=1}^{\infty}$ satisfies the same recurrence relation as does $\{y_n\}_{n=1}^{\infty}$. Since $y_1 = x_1^2$ and $y_2 = x_2^2$, it easily follows by induction that $y_n = x_n^2$ for all $n \geq 1$. Thus, $k = 3$ is a solution of the problem, and we are done.

Remark. Using (2), we can prove that $x_n = f_{2n+1}$ for all $n \geq 1$, where $\{f_n\}$ is the Fibonacci sequence.

Next we move to the February 2006 number of the *Corner* and solutions from readers to problems of the 2003 Vietnamese Mathematical Olympiad, given [2006 : 25–27].

3. Find all polynomials $P(x)$ with real coefficients, satisfying the relation

$$(x^3 + 3x^2 + 3x + 2)P(x-1) \;=\; (x^3 - 3x^2 + 3x - 2)P(x)$$

for every real number x.

Solution by Michel Bataille, Rouen, France.

Let P be a real polynomial satisfying the given condition; that is,

$$(x+2)(x^2+x+1)P(x-1) \;=\; (x-2)(x^2-x+1)P(x)\,. \tag{1}$$

Since the polynomials $x+2$, x^2+x+1, $x-2$, and x^2-x+1 are irreducible over $\mathbb{R}$, we see that $P(x)$ is divisible by $(x+2)(x^2+x+1)$. Thus,

$$P(x) \;=\; (x+2)(x^2+x+1)Q(x) \tag{2}$$

for some real polynomial $Q(x)$, and (1) yields

$$P(x-1) \;=\; (x-2)(x^2-x+1)Q(x)\,. \tag{3}$$

Taking $x = 2$ in (3) gives $P(1) = 0$, which implies that $P(x)$ is divisible by $x - 1$. Since $P(-2) = 0$ (in view of (2)), taking $x = -1$ in (3) gives $Q(-1) = 0$. Therefore, $Q(x)$ is divisible by $x + 1$, and then so is $P(x)$. Lastly, taking $x = 1$ in (1) gives $P(0) = 0$, so that $P(x)$ is divisible by x as well. Summing up, we see that $P(x) = (x+2)(x+1)x(x-1)(x^2+x+1)S(x)$ for some real polynomial $S(x)$.

Now, substituting into (1), we obtain $S(x) = S(x - 1)$ for all $x \in \mathbb{R}$, and an immediate induction shows that $S(n) = S(0)$ for all non-negative integers n. Thus, the polynomial $S(x) - S(0)$ has infinitely many roots. It follows that $S(x)$ is a constant polynomial.

Conversely, substituting $P(x) = k(x + 2)(x + 1)x(x - 1)(x^2 + x + 1)$ in (1) leads to an identity, for all $k \in \mathbb{R}$.

In conclusion, the solutions of the problem are the polynomials of the form $P(x) = k(x + 2)(x + 1)x(x - 1)(x^2 + x + 1)$, where $k \in \mathbb{R}$.

4. Let $P(x) = 4x^3 - 2x^2 - 15x + 9$ and $Q(x) = 12x^3 + 6x^2 - 7x + 1$.

(i) Prove that each of these polynomials has three distinct real roots.

(ii) Let α and β be the greatest roots of $P(x)$ and $Q(x)$, respectively. Prove that $\alpha^2 + 3\beta^2 = 4$.

Solution by Michel Bataille, Rouen, France.

(i) The polynomial P is a continuous function, and it is easily checked that $P(-2) < 0$, $P(-\frac{15}{8}) > 0$, $P(0) > 0$, $P(1) < 0$, and $P(\frac{15}{8}) > 0$. Hence, P has three distinct roots α, α_1, and α_2 satisfying

$$\alpha \in \left(1, \tfrac{15}{8}\right), \quad \alpha_1 \in (0, 1), \quad \alpha_2 \in \left(-2, -\tfrac{15}{8}\right). \tag{1}$$

Similarly, Q has three distinct roots, β, β_1, and β_2 such that

$$\beta \in \left(\tfrac{1}{3}, 1\right), \quad \beta_1 \in \left(0, \tfrac{1}{3}\right), \quad \beta_2 \in (-2, -1). \tag{2}$$

(ii) A polynomial $S(x)$ whose roots are exactly α^2, α_1^2, and α_2^2 is readily obtained by taking S such that $S(x^2) = -P(x) \cdot P(-x)$. Here, since

$$\begin{aligned} S(x^2) &= -\big(9 - 2x^2 + x(4x^2 - 15)\big)\big(9 - 2x^2 - x(4x^2 - 15)\big) \\ &= x^2(4x^2 - 15)^2 - (9 - 2x^2)^2, \end{aligned}$$

we obtain $S(x) = 16x^3 - 124x^2 + 261x - 81$. Similarly, the roots of the polynomial

$$T(x) = 144x^3 - 204x^2 + 37x - 1$$

are β^2, β_1^2, and β_2^2.

Now, transforming $T(x)$ through the relation $y = 4 - 3x$ (that is, substituting $x = (4-y)/3$ in $T(x)$) leads to $T\big((4-y)/3\big) = -\frac{1}{3}S(y)$, which shows that $\{\alpha^2, \alpha_1^2, \alpha_2^2\} = \{4-3\beta^2, 4-3\beta_1^2, 4-3\beta_2^2\}$. Furthermore, from (1) and (2),

$$\alpha^2 \in \left(1, \tfrac{225}{64}\right), \quad \alpha_1^2 \in (0, 1), \quad \alpha_2^2 \in \left(\tfrac{225}{64}, 4\right),$$

so that $\alpha_1^2 < \alpha^2 < \alpha_2^2$ and

$$4 - 3\beta^2 \in \left(1, \tfrac{11}{3}\right), \quad 4 - 3\beta_1^2 \in \left(\tfrac{11}{3}, 4\right), \quad 4 - 3\beta_2^2 \in (-8, 1),$$

so that $4 - 3\beta_2^2 < 4 - 3\beta^2 < 4 - 3\beta_1^2$. The desired result, $\alpha^2 = 4 - 3\beta^2$, follows.

6. Let f be a function defined on the set of real numbers $\mathbb{R}$, taking values in $\mathbb{R}$, and satisfying the condition $f(\cot x) = \sin 2x + \cos 2x$ for every x belonging to the open interval $(0, \pi)$. Find the least and the greatest values of the function $g(x) = f(x) \cdot f(1 - x)$ on the closed interval $[-1, 1]$.

Solution by Michel Bataille, Rouen, France.

We show that g has a minimum value of $4 - \sqrt{34}$ and a maximum value of $1/25$ on $[-1, 1]$.

For $x \in (0, \pi)$, we have

$$\begin{aligned} f(\cot x) &= \sin 2x + \cos 2x = 2 \sin x \cos x + \cos^2 x - \sin^2 x \\ &= \sin^2 x(2 \cot x + \cot^2 x - 1) = \frac{\cot^2 x + 2 \cot x - 1}{\cot^2 x + 1}. \end{aligned}$$

Hence, for all $x \in \mathbb{R}$,

$$f(x) = f\big(\cot(\cot^{-1}(x))\big) = \frac{x^2 + 2x - 1}{x^2 + 1}.$$

Since $g\left(\frac{1}{2} + h\right) = f\left(\frac{1}{2} + h\right) f\left(\frac{1}{2} - h\right) = g\left(\frac{1}{2} - h\right)$ for all real h, it is sufficient to study the values of $g\left(\frac{1}{2} + h\right)$ for $h \in \left[0, \frac{3}{2}\right]$. An easy computation gives

$$g\left(\frac{1}{2} + h\right) = \frac{16h^4 - 136h^2 + 1}{16h^4 + 24h^2 + 25} = 1 - 8\phi(h^2),$$

where ϕ is defined on $\left[0, \frac{9}{4}\right]$ by $\phi(x) = \dfrac{20x + 3}{16x^2 + 24x + 25}$. Since the derivative $\phi'(x)$ has the same sign as $-80x^2 - 24x + 107$, it follows that ϕ reaches its maximum on $\left[0, \frac{9}{4}\right]$ at $x_0 = \dfrac{\sqrt{34}}{5} - \dfrac{3}{20}$ with $\phi(x_0) = \dfrac{20}{32x_0 + 24} = \dfrac{\sqrt{34} - 3}{8}$ and its minimum at 0 with $\phi(0) = \dfrac{3}{25}$. Thus, the extreme values of $g\left(\frac{1}{2} + h\right)$ are $1 - 8\phi(x_0) = 4 - \sqrt{34}$ (minimum) and $1 - 8\phi(0) = \frac{1}{25}$ (maximum).

7. Let α be a real number, $\alpha \neq 0$. Consider the sequence of real numbers $\{x_n\}$, $n = 1, 2, 3, \ldots$, defined by $x_1 = 0$ and $x_{n+1}(x_n + \alpha) = \alpha + 1$ for $n = 1, 2, 3, \ldots$.

(i) Find the general term of the sequence $\{x_n\}$.

(ii) Prove that the sequence $\{x_n\}$ has a finite limit when $n \to +\infty$. Find this limit.

Solved by Houda Anoun, Bordeaux, France; and Mohammed Aassila, Strasbourg, France. We give the solution of Anoun, modified by the editor.

For convenience , let $\beta = -\alpha - 1$. Then $\beta \neq -1$, and the given recurrence relation becomes $x_{n+1}(\beta + 1 - x_n) = \beta$.

Let $u_n = \dfrac{1}{1 - x_n}$ for each n. Then $u_1 = 1$ and for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} u_{n+1} &= \frac{1}{1 - x_{n+1}} = \frac{\beta + 1 - x_n}{(\beta + 1 - x_n) - x_{n+1}(\beta + 1 - x_n)} \\ &= \frac{\beta + 1 - x_n}{\beta + 1 - x_n - \beta} = \frac{\beta + 1 - x_n}{1 - x_n} = 1 + \beta u_n \,. \end{aligned}$$

Case 1. $\beta = 1$.

Then $u_{n+1} = 1 + u_n$. Thus, u_n is an arithmetic sequence, and we have $u_n = n$ for $n = 1, 2, 3, \ldots$. Consequently, the general term of the original sequence $\{x_n\}$ is $x_n = 1 - \dfrac{1}{u_n} = 1 - \dfrac{1}{n}$, and $\lim\limits_{n\to\infty} x_n = 1$.

Case 2. $\beta \neq 1$.

Let $v_n = 1 + (\beta - 1)u_n$. Then $v_1 = \beta$ and for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} v_{n+1} &= 1 + (\beta - 1)u_{n+1} = 1 + (\beta - 1)(1 + \beta u_n) \\ &= \beta + (\beta - 1)\beta u_n = \beta v_n \,. \end{aligned}$$

Thus, $\{v_n\}$ is a geometric sequence. For $n = 1, 2, 3, \ldots$, we have $v_n = \beta^n$. Then $u_n = \dfrac{v_n - 1}{\beta - 1} = \dfrac{\beta^n - 1}{\beta - 1}$, and the general term of the original sequence $\{x_n\}$ is

$$x_n = 1 - \frac{1}{u_n} = 1 - \frac{\beta - 1}{\beta^n - 1} = 1 + \frac{\alpha + 2}{(-\alpha - 1)^n - 1} \,.$$

If $|\beta| < 1$, then $\lim\limits_{n\to\infty} \beta^n = 0$, and hence $\lim\limits_{n\to\infty} x_x = 1 - \dfrac{\beta - 1}{-1} = \beta$. If $|\beta| > 1$, then $\lim\limits_{n\to\infty} |\beta|^n = \infty$, and hence $\lim\limits_{n\to\infty} x_n = 1$.

In summary, the general term is

$$x_n = \begin{cases} 1 - \dfrac{1}{n}\,, & \text{if } \alpha = -2, \\ 1 + \dfrac{\alpha + 2}{(-\alpha - 1)^n - 1}\,, & \text{if } \alpha \neq -2, \end{cases}$$

and the sequence $\{x_n\}$ has a finite limit in all cases.

10. For each integer $n > 1$, denote by s_n the number of permutations $(a_1, a_2, \ldots, a_n)$ of the first n positive integers such that each permutation satisfies the condition $1 \leq |a_k - k| \leq 2$ for $k = 1, 2, \ldots, n$. Prove that $1.75 \cdot s_{n-1} < s_n < 2 \cdot s_{n-1}$ for all integers $n > 6$.

Solution by Mohammed Aassila, Strasbourg, France.

Let n be an integer greater than 6. It can be shown (by induction, for example) that

$$s_n = s_{n-2} + s_{n-4} + 2(s_{n-3} + s_{n-4} + \cdots + s_0) .$$

Then, using this to compute s_{n-1}, we see that

$$s_n = s_{n-1} + s_{n-2} + s_{n-3} + s_{n-4} - s_{n-5} . \qquad (1)$$

Replacing n by $n-1$ yields

$$s_{n-1} = s_{n-2} + s_{n-3} + s_{n-4} + s_{n-5} - s_{n-6} . \qquad (2)$$

Solving (2) for $s_{n-2} + s_{n-3}$ and substituting in (1), we get

$$s_n = 2s_{n-1} - 2s_{n-5} + s_{n-6} . \qquad (3)$$

Since $s_{n-5} > s_{n-6}$, this shows us that $s_n < 2s_{n-1}$, which establishes the right hand inequality.

To prove the left hand inequality, we proceed by induction. It can be directly checked that $s_1 = 0$, $s_2 = 1$, $s_3 = 2$, $s_4 = 4$, and $s_5 = 7$, which means that $\frac{7}{4}s_{n-1} \le s_n$ for $n = 2$, 3, 4, and 5, with strict inequality unless $n = 5$. We will assume that $\frac{7}{4}s_{n-1} \le s_n$ for $n = 2, 3, \ldots, k$ for some integer $k \ge 5$, with strict inequality unless $n = 5$. Using this hypothesis four times successively on (3), we get

$$\begin{aligned} s_{k+1} &= 2s_k - 2s_{k-4} + s_{k-5} > 2s_k - 2s_{k-4} \\ &> 2s_k - 2\left(\tfrac{4}{7}\right)^4 s_k = \tfrac{4290}{2401}s_k > \tfrac{7}{4}s_{n-1} , \end{aligned}$$

which establishes the induction.

Next we look at a solution to one of the problems of the XXIX Russian Mathematical Olympiad, V (Final) Round — 10th Form given [2005 : 27–28].

1. (N. Agakhanov) Let M be a set containing 2003 different positive real numbers, such that for any 3 different elements a, b, c from M the number $a^2 + bc$ is rational. Prove that it is possible to choose a natural number n such that for each a from M the number $a\sqrt{n}$ is rational.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

We present the proof in several steps.

Step 1. If a, $b \in M$ and $a \neq b$, then $a(a+b)$ is rational.

Choose distinct elements c, $d \in M$ different from a and b. Then $a(a+b) = (a^2+bc) + (d^2+ab) - (d^2+bc)$, which is rational.

Step 2. If a, $b \in M$, then b/a is rational.

If $a = b$, then $\dfrac{b}{a} = 1$, which is clearly rational. On the other hand, if $a \neq b$, then $\dfrac{b}{a} = \dfrac{b(a+b)}{a(a+b)}$ is rational by virtue of Step 1.

Step 3. If $a \in M$, then a^2 is rational.

Choose b in M different from a. Then $a(a+b) = a^2\left(1+\frac{b}{a}\right)$. Since $a(a+b)$ and $1+\frac{b}{a}$ are both rational, so is a^2.

Step 4. If $a \in M$, then there exists a positive integer n and a rational number q such that $a = q\sqrt{n}$.

Since a^2 is rational, we have $a^2 = \frac{r}{s}$, where r and s are positive integers. Then $a = \sqrt{\frac{r}{s}} = \frac{1}{s}\sqrt{rs}$. Take $n = rs$ and $q = \frac{1}{s}$.

Step 5. We now complete the proof.

Fix any element $f \in M$. Then $f = q\sqrt{n}$, where n is a positive integer and q is rational. In view of Step 2, if a is any element of M, then $\frac{a}{f} = q_1$ (rational). Hence, $a = q_1 f = q_1 q\sqrt{n}$. Thus, $a\sqrt{n} = q_1 q n$, which is rational.

And next, the one solution on file from readers for problems of the XXIX Russian Mathematical Olympiad, V Final Round — 11th Form given [2006 : 27–28].

1. (N. Agakhanov, A. Golovanov, V. Senderov) Let α, β, γ, and τ be positive numbers such that, for all x,

$$\sin\alpha x + \sin\beta x = \sin\gamma x + \sin\tau x\,.$$

Prove that $\alpha = \gamma$ or $\alpha = \tau$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

We need only assume $\alpha + \beta \neq 0$.

Differentiating the given identity three times, we obtain

$$\begin{aligned} \alpha\cos\alpha x + \beta\cos\beta x &= \gamma\cos\gamma x + \tau\cos\tau x\,, \\ \alpha^3\cos\alpha x + \beta^3\cos\beta x &= \gamma^3\cos\gamma x + \tau^3\cos\tau x\,. \end{aligned}$$

In particular, when $x = 0$, we have

$$\alpha + \beta = \gamma + \tau\,, \tag{1}$$

$$\alpha^3 + \beta^3 = \gamma^3 + \tau^3\,. \tag{2}$$

Cubing both sides of (1), we obtain

$$\alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta) = \gamma^3 + \tau^3 + 3\gamma\tau(\gamma+\tau)\,;$$

hence, $\alpha\beta = \gamma\tau$.

Consequently,

$$(\alpha-\gamma)(\alpha-\tau) = \alpha^2 - (\gamma+\tau)\alpha + \gamma\tau = \alpha^2 - (\alpha+\beta)\alpha + \alpha\beta = 0\,.$$

Therefore, $\alpha = \gamma$ or $\alpha = \tau$.

The next block of solutions from readers are for problems of the Romanian Mathematical Olympiad 9^{th} Grade, given [2006 : 85].

1. Find positive integers a and b such that, for every x, $y \in [a, b]$, we have $\frac{1}{x} + \frac{1}{y} \in [a, b]$.

Solved by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Matti Lehtinen, National Defence College, Helsinki, Finland. We give the solution by Díaz-Barrero.

Let x, $y \in [a, b]$. From $a \leq x$, $y \leq b$, we have $\frac{1}{b} \leq \frac{1}{x}$, $\frac{1}{y} \leq \frac{1}{a}$ and $\frac{2}{b} \leq \frac{1}{x} + \frac{1}{y} \leq \frac{2}{a}$. Since $a \leq \frac{1}{x} + \frac{1}{y} \leq b$, we must have $a \leq \frac{2}{b}$ and $\frac{2}{a} \leq b$, which yields $ab = 2$. Since a and b are integers, the required interval is [1, 2].

2. An integer $n \geq 2$ is called *friendly* if there exists a family A_1, A_2, ..., A_n of subsets of the set $\{1, 2, \ldots, n\}$ such that:

(i) $i \notin A_i$ for every $i \in \{1, 2, \ldots, n\}$;

(ii) $i \in A_j$ if and only if $j \notin A_i$, for every distinct i, $j \in \{1, 2, \ldots, n\}$;

(iii) $A_i \cap A_j$ is non-empty for every i, $j \in \{1, 2, \ldots, n\}$.

Prove: (a) 7 is a friendly number, and (b) n is friendly if and only if $n \geq 7$.

Solution by Matti Lehtinen, National Defence College, Helsinki, Finland.

The table

	A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	−	−	−	−	+	+	+
2	+	−	+	−	−	−	+
3	+	−	−	+	−	+	−
4	+	+	−	−	+	−	−
5	−	+	+	−	−	+	−
6	−	+	−	+	−	−	+
7	−	−	+	+	+	−	−

(+ indicates membership, − non-membership in A_i) shows that 7 is friendly. For any $n > 7$, taking A_1, ..., A_7 from the table and $A_k = \{1, 2, \ldots, 7\}$ for $8 \leq k \leq n$, we get a system of sets showing that n is friendly. It remains to show that no n with $2 \leq n \leq 6$ is friendly. Assume, on the contrary, that some n, $2 \leq n \leq 6$, is friendly and that A_1, ..., A_n are the subsets involved in the definition of friendliness. Assume that A_1, say, is the set having the least number of elements among these sets. Assume there is only one element, say 2, in A_1. By (iii), $A_1 \cap A_2 = \{2\}$, which contradicts (i). This rules out the friendliness of 2. Assume then that A_1 has just two elements, say 2 and 3. Then, by (iii), 2 must be in A_3 and 3 must be in A_2, in violation of (ii). This rules out $n = 3$, and shows that every set A_j has to have at least 3 elements. Now (ii) implies that in a membership table like the one above, the number of $+$'s has to be equal to the number of $-$'s outside the main diagonal. For $n = 4$, $n = 5$, and $n = 6$ the number of $+$'s must be 6, 10, and 15, respectively, and these numbers clearly are less than $3n$ in each of the cases.

3. Prove that the mid-points of the altitudes of a triangle are collinear if and only if the triangle is right.

Solution by Matti Lehtinen, National Defence College, Helsinki, Finland.

Consider $\triangle ABC$, with $\angle C = 90°$. Then AC and BC are two of its altitudes. The line connecting their mid-points bisects every line segment connecting C and AB. Now let C be the largest angle in ABC, $C < 90°$. The feet D and E of the altitudes from A and B are on the segments BC and AC. Thus, the distances of the mid-points P and Q of AD and BE from AB is less than the distance of the mid-point S of the altitude CF. Hence, P and Q are in the half-plane determined by the parallel to AB through S. Therefore, P, Q and S are not collinear. Finally, let $C > 90°$. In this case, the feet D and E of the altitudes dropped from A and B lie on the extensions to BC and AC. Their mid-points P and Q now lie farther away from AB than the mid-point S of the altitude CF. Again, P and Q are both in one of the half-planes determined by the parallel to AB through S. Thus, P, Q, and S are not collinear.

4. Let P be a plane. Prove that there exists no function $f : P \to P$ such that for every convex quadrilateral $ABCD$, the points $f(A)$, $f(B)$, $f(C)$, $f(D)$ are the vertices of a concave quadrilateral.

Solution by Matti Lehtinen, National Defence College, Helsinki, Finland.

Assume such a function f exists. Take any convex pentagon $ABCDE$. Since any four of its five vertices are the vertices of a convex quadrilateral, the convex hull of $\{f(A), \ldots, f(E)\}$ has to be a triangle. Assume the vertices are $f(A)$, $f(B)$, $f(C)$. Since no three of the images can be collinear, $f(D)$ and $F(E)$ are distinct interior points of $f(A)f(B)f(C)$. The lines $f(A)f(D)$, $f(B)f(D)$, $f(C)f(D)$ divide $f(A)f(B)f(C)$ into six triangles, and $f(E)$ is an interior point of one of these (again, no three of the five images can be collinear). It is easy to see that $f(E)$, $f(D)$, and some two of the vertices of $f(A)f(B)f(C)$ are vertices of a convex quadrilateral, a contradiction.

That completes this number of the *Corner*. I need your nice solutions and generalizations sent in within a few months, particularly now that the backlog is cleared up and we are looking at using your submissions within a year.