

SKOLIAD No. 100

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Please send your solutions to the problems in this edition by **1 September, 2007**. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Nos questions proviennent ce mois-ci du Concours de l'Association Mathématique du Québec 2005 (niveau secondaire). Nous remercions Véronique Hussin, Université de Montréal qui s'occupe des concours de l'AMQ du secondaire.

Concours de l'Association Mathématique du Québec (niveau secondaire) 3 février 2005

1. (Le robot et les pommes.) Une caisse de bois est séparée en 9 compartiments comme indiqué sur le dessin. Un ingénieur a programmé un robot pour qu'il remplisse la caisse de pommes par paquets de quatre en laissant tomber une pomme dans chaque compartiment de façon à former un carré 2×2 .

6	16	10
11	28	17
5	12	7

Est-il possible pour le robot d'aboutir à la configuration à droite à partir d'une caisse vide ?

2. (Huit carrés dans un rectangle.) Diviser un rectangle de longueur égale à 9 cm et de largeur égale à 3 cm en huit carrés.

3. (Une étonnante distribution.) Une distribution statistique est composée de 10 nombres naturels : $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. Lorsqu'ils sont placés en ordre croissant, ces nombres nous donnent en fait la distribution suivante : $x_1, x_2, x_3, x_4, x_5, y_5, y_4, y_3, y_2, y_1$. Nous avons plusieurs informations :

- (1) Les couples $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ et (x_5, y_5) , sont tous sur la droite d d'équation $y = -2x + 24$.
- (2) La moyenne de cette distribution est 9,4.
- (3) La médiane et le mode ont tous deux la même valeur.
- (4) Les nombres x_3 et x_4 sont consécutifs.
- (5) Le premier membre de la distribution vaut 1.
- (6) La droite d croise la parabole d'équation $y = \frac{1}{2}x^2 + 8x - 8$ au point (x_2, y_2) .

Trouver les valeurs de la distribution originale $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. Suggestion : La médiane est le nombre tel que 50% des observations sont plus petites ou égales à ce nombre et 50% supérieures ou égales. Le mode est la valeur qui est observée le plus souvent.

4. (La belle somme de Gilbert.) Considérons les 6 façons possibles de permuter (c'est-à-dire mélanger) les chiffres du nombre 123 et additionnons le tout. La somme trouvée s'écrit $123 + 132 + 213 + 231 + 312 + 321 = 1332$.

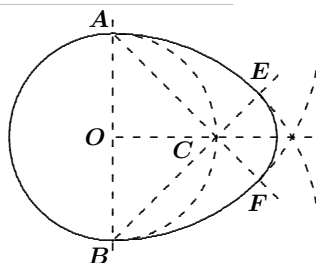
Quel résultat aurions-nous obtenu si nous avions fait la somme des 5040 façons de permuter les chiffres du nombre 1234567 ?

5. (Le voyage à Québec.) Juliette et Philippe partent en même temps et parcourent les 250 km qui séparent Montréal de Québec dans deux voitures identiques. Philippe parcourt la première moitié du trajet à 80 km/h et la seconde moitié à 120 km/h. En fait, il arrive en même temps que Juliette qui a roulé tout le long à vitesse constante. La consommation d'essence de ce type de voiture dépend de la vitesse du véhicule. Elle est donnée par la formule $c = 10 + \frac{v}{20}$, où v est la vitesse en km/h et c la consommation en litres par 100 km. Sachant que ce jour-là, le litre d'essence vaut 0,80\$, combien ont-ils dépensé ensemble pour le voyage ?

6. (Les âges multiples.) Du 21 août 1989 au 7 mai (inclusivement) 1990, Jean a eu 5 fois l'âge de sa fille Claire. Du 8 mai au 20 août (inclusivement) 1992, Jean a eu 4 fois l'âge de sa fille. Trouver la date de naissance de chacun.

Note : par âge, on entend la définition usuelle qui est le nombre d'années complètes écoulées depuis la naissance.

7. (La poule géomètre.) Une figure plane en forme d'œuf est délimitée par quatre arcs de cercles désignés par \widehat{AB} , \widehat{BF} , \widehat{FE} et \widehat{EA} mis bout à bout de la façon indiquée par la figure à droite. Sachant que le rayon AO est de longueur 1, déterminer l'aire de la figure.



Contest of the Mathematical Association of Quebec (Secondary Level) February 3, 2005

1. (The robot and the apples.) A wooden case is separated into 9 compartments as in the drawing. An engineer has programmed a robot to fill the case with apples by dropping four apples at a time, one into each compartment of a 2×2 square.

Is it possible for the robot to finish with the configuration shown if it starts with an empty case?

6	16	10
11	28	17
5	12	7

2. (Eight squares in a rectangle.) Divide a rectangle of length 9 cm and width 3 cm into eight squares.

3. (An astonishing distribution.) A statistical distribution is composed of 10 natural numbers: $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. When placed in increasing order, these numbers are: $x_1, x_2, x_3, x_4, x_5, y_5, y_4, y_3, y_2, y_1$. We also have the following information:

- (1) The couples $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and (x_5, y_5) are all on the line d with equation $y = -2x + 24$.
- (2) The mean of the distribution is 9.4.
- (3) The median and the mode are both the same value.
- (4) The numbers x_3 and x_4 are consecutive.
- (5) The first member of the distribution is 1.
- (6) The line d crosses the parabola having equation $y = \frac{1}{2}x^2 + 8x - 8$ at the point (x_2, y_2) .

Find the values of the original distribution $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. Note: The median is the number such that 50% of the observations are less than or equal to the number and 50% are greater than or equal. The mode is the value which is repeated most often.

4. (Gilbert's beautiful sum.) Consider the 6 different numbers obtained by permuting (mixing) the digits of the number 123. The sum of these numbers is $123 + 132 + 213 + 231 + 312 + 321 = 1332$.

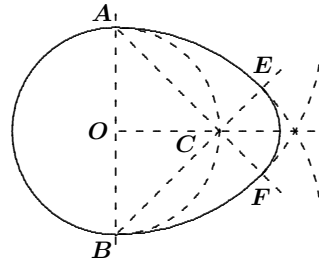
What result would be found if we summed the 5040 different numbers obtained by permuting the digits of the number 1234567?

5. (The trip to Quebec.) Julia and Phillip leave at the same time and cross the 250 km that separate Montreal and Quebec in two identical cars. Phillip does the first half of the trip at 80 km/h and the second half at 120 km/h. He arrives at the same time as Julia, who travelled at a constant speed the whole trip. The fuel consumption for that type of car depends on the speed of the vehicle. It is given by the formula $c = 10 + \frac{v}{20}$, where v is the speed in km/h and c the consumption in liters per 100 km. On that particular day, one litre of gasoline cost \$0.80. How much did they spend on gas altogether for the trip?

6. (Age multiples.) From August 21, 1989 to May 7, 1990 (inclusive), John was 5 times as old as his daughter Claire. From May 8, 1992 to August 20, 1992 (inclusive), John was 4 times as old as his daughter. Find the date of birth of each of them.

Note: By age, we mean, as usual, the number of complete years that have passed since birth.

7. (The geometric hen.) An egg-shaped figure in the plane is composed of four arcs of circles, designated by \widehat{AB} , \widehat{BF} , \widehat{FE} , and \widehat{EA} , put end to end as indicated in the figure. Knowing that the radius AO has length 1, determine the area of the figure.



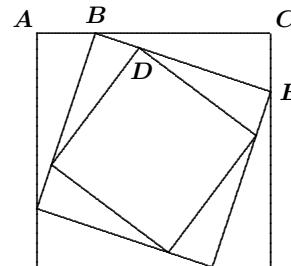
Next we give the solutions to the team round of the fifth annual CNU contest for high school students, run by Ron Persky at Christopher Newport University [2006 : 258–260].

1. Mr. Smith pours a full cup of coffee and drinks $\frac{1}{2}$ of it, deciding it is too strong and needs some milk. So he fills the cup with milk, stirs it, and tastes again, drinking another $\frac{1}{4}$ cup. Once again he fills the cup with milk, stirs it, and finds that this is just as he likes it. What ratio $\frac{\text{amount of coffee}}{\text{amount of milk}}$ does Mr. Smith like?

Solution by Anna Beaudin, Montreal, PQ.

After he drinks another $\frac{1}{4}$ cup, only half of the $\frac{3}{4}$ cup that remains is coffee. This means that $\frac{3}{8}$ is coffee, the rest being milk. Hence, the ratio of coffee to milk is $\frac{3}{5}$.

2. You have three inscribed squares, with the corners of each inner square at the $\frac{1}{4}$ point along the sides of its outer square. (Thus, for example, $AB = \frac{1}{4}AC$ and $BD = \frac{1}{4}BE$.) The area of the largest square is 64 cm^2 . What is the area of the smallest square?



Solution by the editor.

By the Theorem of Pythagoras, we have $BC^2 + CE^2 = BE^2$. Thus, $BE^2 = \frac{9}{16}AC^2 + \frac{1}{16}AC^2 = \frac{5}{8}AC^2$. This means that at each stage, the area of the next smaller square is $\frac{5}{8}$ times the area of the current square. Therefore, the area of the smallest square is $(\frac{5}{8})^2 \cdot 64 = 25 \text{ m}^2$.

3. Solve the equation $\cos 2x = \cos x$ for $0 \leq x < 2\pi$.

Solution by the editor.

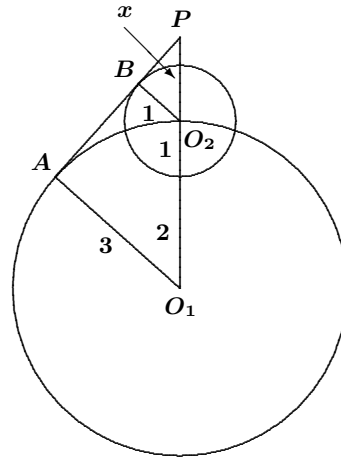
Since $\cos 2x = 2\cos^2 x - 1$, the given condition is equivalent to $2\cos^2 x - \cos x - 1 = 0$, a quadratic equation in the variable $\cos x$. Factoring

gives $(2 \cos x + 1)(\cos x - 1) = 0$, which yields $\cos x = 1$ or $\cos x = -\frac{1}{2}$. Then, since $0 \leq x < 2\pi$, we have $x \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$.

4. The centre of a circle of radius 1 cm is on the circumference of a circle of radius 3 cm. How far (in cm) from the centre of the big circle do the common tangents of the two circles meet?

Solution by the editor.

Let O_1 and O_2 be the centres of the circles of radii 3 and 1, respectively. Let P be the point where the common tangents of the two circles meet. Let A and B be the points of contact of one of the common tangents with the circles of radii 3 and 1, respectively, as shown. Let x denote the length of O_2P . By similar triangles, we have $x : 1 = (x + 3) : 3$, which simplifies to $x = \frac{3}{2}$. Therefore, the distance from the centre of the big circle to the point where the common tangents meet is $PO_1 = \frac{9}{2}$.



5. One root of $2hx^2 + (3h - 6)x - 9 = 0$ is the negative of the other. Find the value of h .

Solution by the editor.

Note that $h \neq 0$, since the given equation has only one root, $x = -\frac{3}{2}$, if $h = 0$. Divide the equation by $2h$ to get

$$x^2 + \frac{3h - 6}{2h}x - \frac{9}{2h} = 0.$$

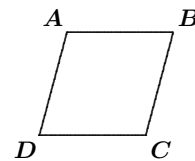
Then the sum of the roots is $-(3h - 6)/(2h)$. However, since one root is the negative of the other, the sum must be 0; that is, $3h - 6 = 0$, which means that $h = 2$.

6. Solve the equation $\sqrt{16x + 1} - 2\sqrt[4]{16x + 1} = 3$.

Solution by the editor.

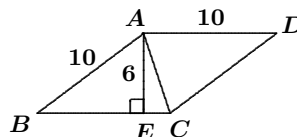
Let $y = \sqrt[4]{16x + 1}$. Then the given equation becomes $y^2 - 2y - 3 = 0$, or $(y - 3)(y + 1) = 0$. Thus, $y = 3$ or $y = -1$. Since $y = \sqrt[4]{16x + 1} > 0$, we cannot have $y = -1$. Therefore, $y = 3$; that is, $\sqrt[4]{16x + 1} = 3$. Then $16x + 1 = 3^4 = 81$, which yields $x = 5$.

7. In the figure $ABCD$, all four sides have length 10 and the area is 60. What is the length of the shorter diagonal AC ?



Solution by the editor.

Drop a perpendicular from the vertex A to the side BC meeting it at E . Since the area of the rhombus is 60, we see that the altitude $AE = 6$. By the Theorem of Pythagoras, we have $BE = 8$, which implies that $EC = 2$. Then the length of the shorter diagonal, AC , can be obtained by applying the Theorem of Pythagoras again:



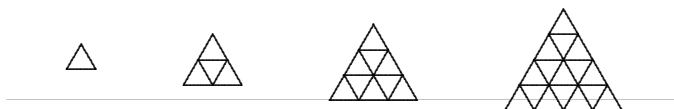
$$AC = \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}.$$

8. A man has 1000 equilateral triangular pieces of mosaic, all of side length 1 cm. He constructs the largest possible mosaic in the form of an equilateral triangle.

- What is the side length of the mosaic?
- How many pieces will he have left over?

Solution by the editor.

(a) Some mosaics in the form of an equilateral triangle that can be constructed from the triangular pieces are shown below.



Notice that the number of pieces in the rows of these mosaics are 1, 3, 5, 7, ... (from top to bottom). If one of these mosaics has sides of length n , then the number of pieces in the bottom row is $2n - 1$, and the total number of pieces in the mosaic is $1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$. (An alternate approach is to notice that each of these mosaics that can be constructed from the triangular pieces uses n^2 pieces for some positive integer n , and has sides of length n .)

The largest value of n such that $n^2 \leq 1000$ is $n = 31$. Thus, the largest possible mosaic has sides of length 31, which is the answer to part (a). The number of pieces left over is then $1000 - 31^2 = 1000 - 961 = 39$, which answers part (b).

That brings us to the end of another issue. Continue sending in your contests and solutions.