SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Tom Leong, Brooklyn, NY, USA has indicated that we omitted a crucial line in his featured solution to problem 3098 [2006 : 532–535]. At the end of the paragraph containing equations (1) and (2), it should have been noted that a solution to the problem follows because $s_k(a_m) = -s_k(b_{n-m-k+2})$, and consequently

$$\sum_{K\in\mathcal{F}} P(K) \;=\; \sum_{m=1}^{n-k+1} s_k(a_m) + \sum_{m=1}^{n-k+1} s_k(b_m) \;=\; 0\,.$$

We apologize for this omission.

2969. [2004 : 368, 371; 2005 : 411–413] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c, d, and r be positive real numbers such that $r = \sqrt[4]{abcd} \geq 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \;\geq\; \frac{4}{(1+r)^2}\,.$$

Remarks by Fan Zhang, Ottawa, ON.

The featured solution to this problem [2005 : 411–413] established the following generalization. For any natural $n > 2$, let a_1, a_2, $\dots$, $a_n > 0$ such that $a_1a_2\cdots a_n = r^n$. Then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \;\geq\; \frac{n}{(1+\sqrt[n]{a_1a_2\cdots a_n})^2}$$

if and only if $r \geq \sqrt{n} - 1$.

For $n = 2$, the editor proved that $r \geq \sqrt{2} - 1$ is necessary but not sufficient, and provided a sufficient condition $r \geq 0.5$. The editor pointed out that the minimum sufficient value of r was not known. We will now show that 0.5 is the minimum sufficient value for r. We will do this by showing that $r \geq 0.5$ is necessary in order that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \;\geq\; \frac{2}{(1+r)^2}$$

when a and b are positive real numbers and $r^2 = ab$.

Proposition. Suppose that $\sqrt{2}-1 < r < 0.5$. Then there exist positive real numbers a and b such that $ab = r^2$ and

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} < \frac{2}{(1+r)^2}\,.$$

In fact, one may take $a = \dfrac{1-r^3-r^2-r}{r^2+2r-1}$ and $b = r^2/a$.

Proof: We first observe that, if $a = \dfrac{1-r^3-r^2-r}{r^2+2r-1}$, then

$$a \;=\; \frac{(1-2r+r^4)/(1-r)}{\big(r+1+\sqrt{2}\big)\big(r+1-\sqrt{2}\big)} \;>\; \frac{r^4/(1-r)}{\big(r+1+\sqrt{2}\big)\big(r+1-\sqrt{2}\big)} \;>\; 0\,.$$

To show that $\dfrac{1}{(1+a)^2} + \dfrac{1}{(1+b)^2} < \dfrac{2}{(1+r)^2}$, we will prove the equivalent statement, $f(r) < 0$, where

$$f(r) = \left(\frac{1}{(1+a)^2} + \frac{1}{\left(1+\dfrac{r^2}{a}\right)^2} - \frac{2}{(1+r)^2}\right)(1+a)^2\left(1+\frac{r^2}{a}\right)^2(1+r)^2a^2\,.$$

Using a computer algebra system, $f(r)$ simplifies to

$$\begin{aligned} f(r) \;&=\; -2ar^2+4a^2r-6a^2r^2+4ar^3-2ar^4+4a^3r-2a^3r^2 \\ &\qquad +a^4r^2-2a^2r^4-r^4+2r^5+r^6-2a^3-a^4 \\ &=\; (r-a)^2\big(2a^2r+a^2r^2-a^2+2ar-2a \\ &\qquad +2a^4r+2ar^3+2ar^2-r^2+r^4+2r^3\big) \\ &=\; (r-a)^2\big[(r^2+2r-1)a^2-2(1-r^3-r^2-r)a \\ &\qquad +(r^2+2r-1)r^2\big]\,. \end{aligned}$$

Substituting $a = \dfrac{1-r^3-r^2-r}{r^2+2r-1}$ and noticing that

$$r-a \;=\; \frac{2r^3+3r^2-1}{r^2+2r-1} \;=\; \frac{(2r-1)(r+1)^2}{r^2+2r-1}\,,$$

we see that

$$\begin{aligned} f(r) \;&=\; \left(\frac{(2r-1)(r+1)^2}{r^2+2r-1}\right)^2(r^2+2r-1)\left[r^2-\left(\frac{1-r^3-r^2-r}{r^2+2r-1}\right)^2\right] \\ &=\; \frac{(2r-1)^2(r+1)^4\big[(2r-1)(r+1)^2(r-1)^2\big]}{\big(r^2+2r-1\big)^3} \;<\; 0\,. \end{aligned}$$

3114. [2006 : 107, 109] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, c be positive real numbers such that

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} \;=\; 2\,.$$

Prove that

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{1}{4c+1} \;\geq\; 1\,.$$

I. *Solution by Alex Remorov, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON.*

Note that for any positive real number x,

$$\frac{1}{4x+1} \;\geq\; \frac{1}{x+1}-\frac{1}{3}\,, \qquad (1)$$

because this inequality is equivalent in succession to

$$\begin{aligned}
\frac{1}{4x+1} &\geq \frac{2-x}{3(x+1)}\,,\\
3x+3 &\geq (2-x)(4x+1)\,,\\
4x^2-4x+1 &\geq 0\,,\\
(2x-1)^2 &\geq 0\,,
\end{aligned}$$

which is obviously true.

Setting $x=a$, b, c in (1) and adding, we obtain

$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{1}{4c+1} \;\geq\; \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}-1 \;=\; 1\,.$$

II. *Generalization by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

We shall prove the generalization that if $x_0, x_1, \ldots, x_n$ are positive real numbers such that $\sum\limits_{k=0}^{n}\frac{1}{x_k+1}=n$, then $\sum\limits_{k=0}^{n}\frac{1}{n^2x_k+1}\geq 1$. The proposed inequality is the special case where $n=2$.

Since the result is obvious when $n=1$, we assume $n>1$. For any real number x,

$$\begin{aligned}
&(n+1)^2(x+1)-(n+1)^2(n^2x+1)+(n^2-1)(n^2x+1)(x+1)\\
&\quad= (n+1)^2(x-n^2x)+(n^2-1)(n^2x+1)(x+1)\\
&\quad= (n^2-1)((n^2x+1)(x+1)-(n+1)^2x)\\
&\quad= (n^2-1)(n^2x^2-2nx+1) \;=\; (n^2-1)(nx-1)^2 \;\geq\; 0\,.
\end{aligned}$$

with equality if and only if $x=1/n$. In particular, for positive x, we may divide by $(n+1)^2(x+1)(n^2x+1)$ and re-arrange terms to get

$$\frac{1}{n^2x+1} \;\geq\; \frac{1}{x+1}-\frac{n^2-1}{(n+1)^2}\,, \qquad (2)$$

with equality if and only if $x = 1/n$.

Finally, apply (2) to $x_0, x_1, \ldots, x_n$ to obtain

$$\sum_{k=0}^{n} \frac{1}{n^2 x_k + 1} \geq \sum_{k=0}^{n} \frac{1}{x_k + 1} - (n+1)\frac{n^2-1}{(n+1)^2} = n - \frac{n^2-1}{n+1} = 1,$$

with equality if and only if $x_0 = x_1 = \cdots = x_n = 1/n$.

Also solved by ARKADY ALT, San Jose, CA, USA; ROY BARBARA, University of Beirut, Beirut, Lebanon; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; JIM BLACK, student, Missouri State University, Springfield, MO, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; CAO MINH QUANG, Nguyen Binh Khiem specialized high school, Vinh Long, Vietnam; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

About half of the solvers used calculus or convexity and Jensen's Inequality. Zhou showed that the result is actually true for $a, b, c \in (-\infty, -1) \cup (-1/4, \infty)$. Several other generalizations were obtained. Benito, Ciaurri, and Fernández proved that if $n \geq 3$ and $a_1, \ldots, a_n$ are positive real numbers such that $\sum_{i=1}^{n} \frac{1}{a_i+1} = 2$, then $\sum_{i=1}^{n} \frac{1}{k^2 a_i+1} \geq 1$, for $k = \frac{n+1}{n-1}$. Their proof is a straightforward generalization of Solution I above. Janous proved that if $n \geq 2$ and $x_1, x_2, \ldots, x_n$ are positive real numbers such that $\sum_{i=1}^{n} \frac{1}{x_i+1} = a$, where $a < n$ is a constant, then $\sum_{i=1}^{n} \frac{1}{bx_i+1} \geq \frac{an}{b(n-a)+a}$ for all constants $b > 1$. The special case when $n = 3$, $a = 2$, and $b = 4$ is the proposed inequality. Quang established the similar result that if $\sum_{i=1}^{n} \frac{1}{x_i+1} \geq 1$, then $\sum_{i=1}^{n} \frac{1}{4x_i+1} \geq \frac{n}{4n-3}$.

3115. [2006 : 107, 109] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let a, b, c, be the lengths of the sides opposite the vertices A, B, C, respectively, in triangle ABC. Prove that

$$\frac{\cos^3 A}{a} + \frac{\cos^3 B}{b} + \frac{\cos^3 C}{c} < \frac{a^2+b^2+c^2}{2abc}.$$

Essentially the same solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let R be the circumradius of $\triangle ABC$. By the Law of Sines, we have

$$\sum_{\text{cyclic}} \left(b^2+c^2-a^2\right)\sin^2 A$$

$$= \sum_{\text{cyclic}} \frac{a^2\left(b^2+c^2-a^2\right)}{4R^2} = \frac{2\left(a^2b^2+b^2c^2+c^2a^2\right)-\left(a^4+b^4+c^4\right)}{4R^2}$$

$$= \frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{4R^2} > 0.$$

Hence,

$$\begin{aligned}\sum_{\text{cyclic}}(b^2+c^2-a^2)\cos^2 A &= \sum_{\text{cyclic}}(b^2+c^2-a^2)(1-\sin^2 A)\\ &< \sum_{\text{cyclic}}(b^2+c^2-a^2) \;=\; a^2+b^2+c^2\,,\end{aligned}$$

which is equivalent to $\sum_{\text{cyclic}}\left(2bc\cos^3 A\right) < a^2+b^2+c^2$. Dividing both sides by $2abc$, the result follows immediately.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, University of Beirut, Beirut, Lebanon; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer. There were also two incorrect solutions.

Both Janous and Zvonaru showed that the given inequality is equivalent to

$$\sum_{\text{cyclic}} a^2(b^2+c^2-a^2)^3 \;<\; 4a^2b^2c^2(a^2+b^2+c^2)\,,$$

and remarked that this is a special case of Crux problem #3116 (by the same proposer). Zvonaru also pointed out that if $\triangle ABC$ is an acute triangle, then the following is a very simple proof of the given inequality:

$$\sum_{\text{cyclic}}\frac{\cos^3 A}{a} < \sum_{\text{cyclic}}\frac{\cos A}{a} = \sum_{\text{cyclic}}\frac{b^2+c^2-a^2}{2abc} = \frac{a^2+b^2+c^2}{2abc}\,.$$

3116. [2006 : 107, 110] *Proposed by Arkady Alt, San Jose, CA, USA.*

For arbitrary real numbers a, b, c, prove that

$$\sum_{\text{cyclic}} a(b+c-a)^3 \;\le\; 4abc(a+b+c)\,.$$

Essentially the same solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; and Joel Schlosberg, Bayside, NY, USA.

$$4abc(a+b+c)-\sum_{\text{cyclic}} a(b+c-a)^3 \;=\; (a^2+b^2+c^2-2ab-2ac-2bc)^2 \;\ge\; 0\,.$$

The equality holds if and only if $a=b$ and $c=0$, $b=c$ and $a=0$, or $c=a$ and $b=0$.

Also solved by ROY BARBARA, University of Beirut, Beirut, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; VEDULA N. MURTY, Dover, PA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer.

3117. [2006 : 107, 110] *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let a, b, c be the lengths of the sides and s the semi-perimeter of $\triangle ABC$. Prove that

$$\sum_{\text{cyclic}} (a+b)\sqrt{ab(s-a)(s-b)} \;\le\; 3abc\,.$$

Essentially the same solution by Mohammed Aassila, Strasbourg, France; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Titu Zvonaru, Comăneşti, Romania.

We have

$$\begin{aligned} abc^2-(a+b)^2(s-a)(s-b) &= abc^2-(a+b)^2\left[\frac{c^2-(a-b)^2}{4}\right] \\ &= \frac{(4ab-(a+b)^2)c^2+(a+b)^2(a-b)^2}{4} \\ &= \tfrac{1}{4}(a-b)^2[(a+b)^2-c^2] \;\ge\; 0\,, \end{aligned}$$

since $c \le a+b$ by the Triangle Inequality. Equality holds if and only if $a=b$. It follows that $(a+b)\sqrt{ab(s-a)(s-b)} \le \sqrt{ab}\sqrt{abc^2} = abc$, with equality if and only if $a=b$. Then

$$\sum_{\text{cyclic}} (a+b)\sqrt{ab(s-a)(s-b)} \;\le\; \sum_{\text{cyclic}} abc \;=\; 3abc\,,$$

with equality if and only if $\triangle ABC$ is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3118. [2006 : 108, 110] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let BE and CF be altitudes of the acute-angled triangle ABC with E on AC and F on AB. Let BK and CL be the interior angle bisectors of $\angle ABC$ and $\angle ACB$, respectively, with K on AC and L on AB. Let I denote the incentre of $\triangle ABC$, and let O denote its circumcentre. Prove that E, F, and I are collinear if and only if K, L, and O are collinear.

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

This nice problem was proposed (but not used) at the 38th IMO held at Mar del Plata (Argentina), in 1997.

We use a result known as the "Theorem of Transversals", which establishes necessary and sufficient conditions for a line which cuts two sides of a triangle to pass through some of the noteworthy points of the triangle. For a detailed discussion of the subject, see [1].

Since $\triangle ABC$ is acute, a necessary and sufficient condition for K, L, and O to be collinear is

$$\frac{BL}{LA}\cdot\sin 2B+\frac{CK}{KA}\cdot\sin 2C \;=\; \sin 2A\,. \tag{1}$$

A necessary and sufficient condition for E, F, and I to be collinear is

$$\frac{BF}{FA}\cdot b+\frac{CE}{EA}\cdot c \;=\; a\,. \tag{2}$$

In our case, since BE and CF are altitudes, we have $BF = a\cdot\cos B$, $FA = b\cdot\cos A$, $CE = a\cdot\cos C$, and $EA = c\cdot\cos A$. Thus, equation (2) takes the form $\frac{\cos B}{\cos A}+\frac{\cos C}{\cos A}=1$; that is,

$$\cos B+\cos C \;=\; \cos A\,. \tag{3}$$

On the other hand, since BK and CL are internal bisectors, the theorem on internal bisectors gives us directly that $\frac{BL}{LA}=\frac{a}{b}$ and $\frac{CK}{KA}=\frac{a}{c}$. Therefore, equation (1) can be written as $\frac{a}{b}\sin 2B+\frac{a}{c}\sin 2C \;=\; \sin 2A$; that is,

$$\cos B\left(\frac{\sin B}{b}\right)+\cos C\left(\frac{\sin C}{c}\right) \;=\; \cos A\left(\frac{\sin A}{a}\right)\,.$$

Using the Law of Sines, we reduce this equation to (3), and we are done.

II. *Solution by Michel Bataille, Rouen, France.*

In areal coordinates relative to $\triangle ABC$, we have

$$I(a,b,c)\,,\quad E(a\cos C,0,c\cos A)\,,\quad \text{and}\quad F(a\cos B,b\cos A,0)\,.$$

Hence, I, E, and F are collinear if and only if

$$\begin{vmatrix} a & a\cos C & a\cos B\\ b & 0 & b\cos A\\ c & c\cos A & 0\end{vmatrix} \;=\; 0\,,$$

which reduces to $\cos A(\cos B+\cos C-\cos A)=0$.

Similarly, from the areal coordinates $O(a\cos A,b\cos B,c\cos C)$, $K(a,0,c)$, and $L(a,b,0)$, we see that O, K, and L are collinear if and only if $\cos B+\cos C-\cos A=0$.

In conclusion, if O, K, and L are collinear, then I, E, and F are also collinear (in any triangle ABC). Conversely, if I, E, and F are collinear and $\angle A\neq 90^\circ$, then O, K, and L are collinear.

References

[1] Francisco Bellot Rosado, *Un théorème peu connu : le théorème des transversales;* Mathématique et Pédagogie, n^o 153, 41–55, 2005.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Demis posed the following question: If the side BC is a constant and the points K, L, and O remain collinear, then

1. *does A lie on an interesting curve?*
2. *do we obtain the maximum or minimum distance of A from BC if and only if $AB = AC$?*

3119. [2006 : 108, 110] *Proposed by Michel Bataille, Rouen, France.*

Let r and s denote the inradius and semi-perimeter, respectively, of triangle ABC. Show that

$$3\sqrt{3}\sqrt{\frac{r}{s}} \;\le\; \sqrt{\tan\left(\tfrac{1}{2}A\right)}+\sqrt{\tan\left(\tfrac{1}{2}B\right)}+\sqrt{\tan\left(\tfrac{1}{2}C\right)} \;\le\; \sqrt{\frac{s}{r}}\,.$$

A combination of nearly identical solutions from Mohammed Aassila, Strasbourg, France; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.

Since $\tan\frac{A}{2} = \frac{r}{s-a}$, $\tan\frac{B}{2} = \frac{r}{s-b}$, and $\tan\frac{C}{2} = \frac{r}{s-c}$, the proposed inequalities may be rewritten as

$$\frac{3\sqrt{3}}{\sqrt{s}} \;\le\; \frac{1}{\sqrt{s-a}}+\frac{1}{\sqrt{s-b}}+\frac{1}{\sqrt{s-c}} \;\le\; \frac{\sqrt{s}}{r}\,.$$

The inequality on the left is an immediate consequence of Jensen's Inequality (which, in the present context, is just the Power Mean Inequality). Specifically, using the convexity of $1/\sqrt{t}$, we obtain

$$\frac{1}{\sqrt{s-a}}+\frac{1}{\sqrt{s-b}}+\frac{1}{\sqrt{s-c}} \;\ge\; 3\cdot\frac{1}{\sqrt{\dfrac{(s-a)+(s-b)+(s-c)}{3}}} \;=\; \frac{3\sqrt{3}}{\sqrt{s}}\,.$$

Equality holds if and only if $\triangle ABC$ is equilateral.

The other inequality is a consequence of the AM–GM Inequality, as follows:

$$\begin{aligned}
&\sqrt{(s-a)(s-b)(s-c)}\left(\frac{1}{\sqrt{s-a}}+\frac{1}{\sqrt{s-b}}+\frac{1}{\sqrt{s-c}}\right)\\
&\qquad= \sqrt{(s-b)(s-c)}+\sqrt{(s-c)(s-a)}+\sqrt{(s-a)(s-b)}\\
&\qquad\le \frac{2s-b-c}{2}+\frac{2s-c-a}{2}+\frac{2s-a-b}{2}\\
&\qquad= s = \frac{\text{Area}(ABC)}{r} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{r}\,,
\end{aligned}$$

which yields the desired result. Once again, equality holds if and only if $\triangle ABC$ is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ÓSCAR CIAURRI, Universidad de La Rioja, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.M. MILOŠEVIĆ, Pranjani, Yugoslavia; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comănești, Romania.

3120. [2006 : 108, 110] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be an isosceles triangle with $AB = BC$, and let F be the mid-point of AC. Let $\alpha = \angle BAX$, where X is a variable point on the ray BF. As long as $\alpha \neq \pi/2$, the reflections of the line BF in BA and XA intersect. Let that point of intersection be denoted by M.

Find $\lim\limits_{\alpha \to \pi/2} |\cos\alpha| \cdot CM$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let $r = AF$ $(= \frac{1}{2}AC)$. We shall see that r is the desired limit. Since line MB is the reflection of BF across BA, and line MX is the reflection of line BF across AX, then A is equidistant from the three sides of $\triangle BMX$. Indeed, if α $(= \angle BAX)$ is sufficiently near $\pi/2$, then r is the inradius when A and M are on the same side of BX, and the exradius otherwise. Let $\theta = \angle BMA = \angle XMA$. Then $\alpha - \theta = \pi/2$ when A is the incentre, and $\alpha + \theta = \pi/2$ when A is the excentre. Either way, we have $|\cos\alpha| = |\sin\theta|$. Next, observe that the line MB is fixed, and as $\alpha \to \pi/2$, the point M goes to infinity along that fixed line and $MA/MC \to 1$. But $|\cos\alpha| = |\sin\theta|$; therefore,

$$\lim_{\alpha \to \pi/2} |\cos\alpha| MC \;=\; \lim_{\alpha \to \pi/2} |\sin\theta| MA \;=\; r \;=\; AF\,.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, University of Beirut, Beirut, Lebanon; ÓSCAR CIAURRI, Universidad de La Rioja, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (two solutions); JOEL SCHLOSBERG, Bayside, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3121. [2006 : 108, 110] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n and r be positive integers. Show that

$$\left(\frac{1}{2^n}\sum_{k=1}^{n}\frac{1}{k}\binom{n-1}{k-1}\left[1-\frac{1}{2^{nr}}\binom{n}{k}^{r}\right]\right)^{r} \;\leq\; \frac{r^r}{(r+1)^{r+1}}\,.$$

Solution by Michel Bataille, Rouen, France.

The proposed inequality is equivalent to

$$\frac{1}{2^n}\sum_{k=1}^{n}\frac{1}{k}\binom{n-1}{k-1}\left[1-\frac{1}{2^{nr}}\binom{n}{k}^r\right] \;\le\; \frac{r}{r+1}\cdot\frac{1}{(r+1)^{1/r}}\,. \qquad (1)$$

Let L denote the left side of (1). Then

$$L \;=\; \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^n}\binom{n}{k}\left[1-\left(\frac{1}{2^n}\binom{n}{k}\right)^r\right]\,.$$

Since $\sum\limits_{k=0}^{n}\binom{n}{k}=2^n$, we have $0<\frac{1}{2^n}\binom{n}{k}<1$ for $k=1,\,2,\,\dots,\,n$.

Using simple calculus, it is easy to show that the function f defined by $f(x)=x(1-x^r)$, for $x\in[0,1]$, attains its absolute maximum M at $x=\dfrac{1}{(r+1)^{1/r}}$. Hence,

$$\begin{aligned} L &= \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{1}{2^n}\binom{n}{k}\right) \;\le\; \frac{1}{n}\sum_{k=1}^{n} M \;=\; M \\ &= f\left(\frac{1}{(r+1)^{1/r}}\right) \;=\; \frac{r}{r+1}\cdot\frac{1}{(r+1)^{1/r}}\,, \end{aligned}$$

and (1) follows.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

The proof featured above actually shows that the given inequality holds for all positive real numbers r. This was pointed out explicitly by Benito, Ciaurri, and Fernández.

3122. [2006 : 108, 111] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $\triangle ABC$ and $\triangle A'B'C'$ have right angles at A and A', respectively, and let h_a and $h_{a'}$ denote the altitudes to the sides a and a', respectively. If $b\ge c$ and $b'\ge c'$, prove that

$$\sqrt{aa'}+2\sqrt{h_a h_{a'}} \;\le\; \sqrt{2}\left(\sqrt{bb'}+\sqrt{cc'}\right)\,.$$

Solution by Michel Bataille, Rouen, France.

Since $ah_a = bc$ and $a'h_{a'} = b'c'$, the proposed inequality may be expressed as

$$\sqrt{aa'}+2\sqrt{\frac{bb'cc'}{aa'}} \;\le\; \sqrt{2}(\sqrt{bb'}+\sqrt{cc'})\,.$$

Squaring and multiplying by aa' gives the equivalent inequality

$$(aa')^2 + 4bb'cc' \ \le \ 2aa'(bb' + cc')\,,$$

which may be rewritten as

$$(aa' - 2bb')(aa' - 2cc') \ \le \ 0\,. \qquad (1)$$

By means of the Cauchy-Schwarz Inequality, we get

$$aa' \ = \ \sqrt{b^2+c^2}\sqrt{(b')^2+(c')^2} \ \ge \ bb' + cc' \ \ge \ 2cc'$$

(the last inequality because $b \ge c$ and $b' \ge c'$). Thus, the second factor of (1) is non-negative.

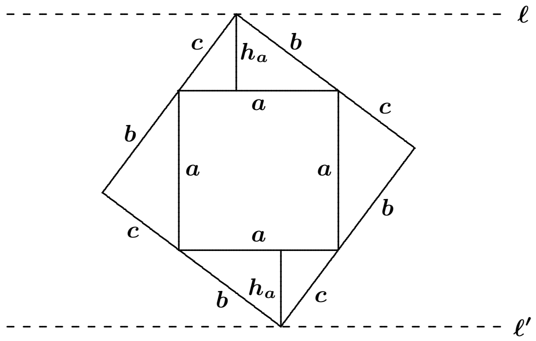
Since $b^2 + c^2 \le 2b^2$ and $(b')^2 + (c')^2 \le 2(b')^2$, we get

$$aa' \ = \ \sqrt{b^2+c^2}\sqrt{(b')^2+(c')^2} \ \le \ \sqrt{2}b \cdot \sqrt{2}b' \ = \ 2bb'\,,$$

and the first factor in (1) is non-positive. The result follows at once.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, University of Beirut, Beirut, Lebanon (2 solutions); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Barbara and the proposer point out that, if the triangles are similar, then the proposed inequality simplifies to $a + 2h_a \le \sqrt{2}(b+c)$. Barbara provided a visualization of this inequality, shown below. Clearly, $a + 2h_a$ is the distance from ℓ to ℓ' and $\sqrt{2}(b+c)$ is the length of the diagonal of the large square.



3123. [2006 : 111] *Proposed by Joe Howard, Portales, NM, USA.*

Let a, b, c be the sides of a triangle. Show that

$$\frac{abc(a+b+c)^2}{a^2+b^2+c^2} \ \ge \ 2abc + \prod_{\text{cyclic}} (b+c-a)\,.$$

Solution by Titu Zvonaru, Comăneşti, Romania.

We prove that the given inequality is true for any three non-negative numbers a, b, and c such that $a^2+b^2+c^2>0$.

For any such a, b, and c, we have

$$\begin{aligned}
&abc(a+b+c)^2-(a^2+b^2+c^2)\Big(2abc+\prod_{\text{cyclic}}(b+c-a)\Big)\\
&= abc\Big(\sum_{\text{cyclic}}a^2+2\sum_{\text{cyclic}}bc\Big)-(a^2+b^2+c^2)\Big(\sum_{\text{cyclic}}(a^2b+a^2c)-\sum_{\text{cyclic}}a^2\Big)\\
&= \sum_{\text{cyclic}}(a^3bc+2ab^2c^2)-\sum_{\text{cyclic}}(a^4b+a^4c+2ab^2c^2-a^5)\\
&= \sum_{\text{cyclic}}a^3(a^2+bc-ab-ac) \;=\; \sum_{\text{cyclic}}a^3(a-b)(a-c) \;\geq\; 0\,,
\end{aligned}$$

by Schur's Inequality. The desired inequality follows immediately. Equality holds if and only if $a=b=c$ or two of the numbers a, b, c are equal and the third is zero.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia; VEDULA N. MURTY, Dover, PA, USA; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3124. [2006 : 109, 111] *Proposed by Joe Howard, Portales, NM, USA.*

Let a, b, c be the sides of $\triangle ABC$ in which at most one angle exceeds $\pi/3$, and let r be its inradius. Show that

$$\frac{\sqrt{3}(abc)}{a^2+b^2+c^2} \;\geq\; 2r\,.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let R and s denote the circumradius and semiperimeter of $\triangle ABC$, respectively. We first use the well-known formulas $abc=4Rrs$ and $a^2+b^2+c^2=2(s^2-r^2-4Rr)$ to write the given inequality as $\sqrt{3}Rs\geq s^2-r^2-4Rr$. This is equivalent to

$$s^2-\sqrt{3}Rs-r(4R+r) \;\leq\; 0\,,$$

or $(s - x_1)(s + x_2) \leq 0$, where

$$x_1 = \frac{\sqrt{3}R + \sqrt{3R^2 + 16Rr + 4r^2}}{2}$$

$$\text{and} \quad x_2 = \frac{-\sqrt{3}R + \sqrt{3R^2 + 16Rr + 4r^2}}{2}.$$

Obviously, $s + x_2 > 0$. Therefore, we just have to prove that $s \leq x_1$.

Now, it is known ([1, section 37, and also p. 256]) that for any triangle satisfying the given condition, we have $s \leq \sqrt{3}(R + r)$. [*Ed:* A proof of this may also be found in Howard's featured solution to #2887 [2004 : 519].] To show that $s \leq x_1$, it then suffices to show that

$$2\sqrt{3}(R + r) \leq \sqrt{3}R + \sqrt{3R^2 + 16Rr + 4r^2}.$$

Using some simple algebra, it is easily seen that this inequality is equivalent to $3(R + 2r)^2 \leq 3R^2 + 16Rr + 4r^2$, or $2r \leq R$, which is a celebrated and well-known result of Euler.

References

[1] A. Bager, *A Family of Goniometric Inequalities*, Univ. Beograd. Publ. El. Fak. Ser. Mat. Fiz. 338–352 (1971), 5–25.

[2] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; VEDULA N. MURTY, Dover, PA, USA; and the proposer whose proof is virtually the same as the one given above. There was also an incomplete solution.

Janous mentioned that triangles which satisfy the described condition were "baptized" as "triangles of Bager's type II" (see [2, 256–261]).