

Pólya's Paragon

Playing Games with Mathematics (Part II)

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The September 2006 Paragon consisted of five challenges that were presented as the opening of a two-part Paragon. Here in Part II, the challenges are repeated and immediately followed by some insights into the mathematical principles beneath these challenges.

1. Sim

Six dots are drawn on a piece of paper to form the vertices of a hexagon. Two players are each assigned a colour. The players take turns joining any two of the dots with a line segment, using their assigned colours. The loser is the player who completes a triangle with three of the original six dots as its vertices and with all three edges the same colour.

Challenge: Prove that there must always be a loser (and a winner).

An outline of a proof by contradiction is provided here. Assume that it is possible to have a tie. Let the original dots be A , B , C , D , E , and F . Consider any one of these dots. Suppose we choose A . Note that there are five segments that can be drawn from A . Select two colours, say blue and red. At least three of the segments from A must be one of these colours, say blue. Again it does not matter which three segments. Suppose that AB , AC , and AE are blue. It follows that none of BC , BE , or CE are blue because otherwise a blue triangle would be formed, thus creating a loser. Aha! That makes triangle BCE a red triangle. Therefore, we have a proof by contradiction that a tie is impossible.

Readers interested in reading more about this and other combinatorial problems may refer to sources such as [3].

2. 31

This mental math game involves a running total which starts at zero. Each player has the choice to add 1, 2, 3, 4, 5, or 6 to the total. Players alternate turns. The winner is the player who is able to bring the total to 31.

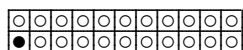
Challenge: Determine a winning strategy. You may choose to play first or second.

This is the easiest of the five challenges. Working backwards from 31, you can determine that a player who gets to 24 should win the game. Continuing backwards, we find that 17, 10, and 3 are winning positions. The winning strategy is to begin with 3 and ensure that you get each of 10, 17, and 24 on your way to 31. A misstep will allow your opponent to claim a winning position.

3. Chomp

Counters are placed in a rectangular grid such that one counter appears in each small rectangle. The counter in the bottom left-hand corner is a different colour than the others. Players take turns selecting one counter. If the counter selected occupies the bottom left-hand corner of a rectangle on the grid, all the counters in that rectangle are removed. The object is to force your opponent to select the differently coloured counter (the one in the bottom left-hand corner).

Challenge: Suppose that you play two games of Chomp in which the boards are $2 \times n$ and $k \times k$, examples of which are shown. Determine a winning strategy in each case. You may choose to play first or second.



This game appears in various sources. My reference is [4], which recommends [1] for further reading on the game. (Coincidentally, a review of the collection [1] will soon appear in *CRUX with MAYHEM*.) We now turn our attention to the winning strategies for the pictured boards. The boards shown above are among those for which winning strategies do exist for the first player.

The solution for the $2 \times n$ board is to remove the single square in the top right-hand corner, thus, producing a symmetric situation in which the first player can copy the moves of the second player until the second player is forced to take the unwanted counter. The $k \times k$ example uses a similar idea: remove the entire figure except for the squares along the left edge and the bottom edge, thus leaving an L-shape of unit width. Observe that the first player can again engage in a copying strategy to ensure victory in the game.

Those of you who enjoy Chomp might want to consider yet another challenge posed in [4]: determine the unique opening move for a winning strategy with a 3×5 board.

4. Fifteen Finesse

The numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 are available for use in this game. Each number can be used only once. Two players alternate turns selecting one of the available numbers. To win the game, a player must obtain exactly three numbers that sum to 15. (Neither a pair of numbers, such as 7 and 8, nor a set of four numbers, such as 1, 3, 5, and 6, constitutes a winning combination.) The game ends in a draw if no player is able to acquire three numbers that sum to 15.

Challenge: Explain the underlying structure of the game, and suggest strategies that may help to win a game.

Does this game sound vaguely familiar? My experience has been that people play this game joyfully as they win, lose, or draw. Can you identify another game with an objective of three-in-a-row that you may win, lose, or

draw while learning the game? Indeed, Tic-Tac-Toe! Where did we get three-in-a-row from the description of the game? Here we have an example of an isomorphism. That is, Fifteen Finesse is isomorphic to playing Tic-Tac-Toe on a magic square board. One such board is provided here for consideration:

8	1	6
3	5	7
4	9	2

This game can be an embarrassment! I suspect that if you recorded the sequence of number selections in early games and transposed them onto the board, you would not be so proud of some of your moves. It's fun to see how one played the game prior to having knowledge of the underlying structure. This game was introduced to me twenty years ago in [2], which has recently been reprinted by the MAA and is to be reviewed in *CRUX with MAYHEM*.

5. A Polynomial in Transition

Consider the polynomial $x^2 + 10x + 20$. Under the conditions below, is it possible to convert this polynomial to $x^2 + 20x + 10$? Justify your answer.

Conditions:

- (i) On each step you may only change the constant term or the coefficient of x (but not both).
- (ii) The change must be an increase of 1 or a decrease of 1.
- (iii) The change must NOT produce a polynomial that can be factored into the form $(x + m)(x + n)$ where m and n are integers. For example, you could not begin by reducing 10 to 9, since $x^2 + 9x + 20 = (x + 5)(x + 4)$.

This challenge, drawn to my attention in a problem solving course in 1988 by Ed Barbeau (see [?]), can be solved readily with an insight. Suppose that we consider the polynomial to be of the form $x^2 + bx + c$. Consider the value $b - c$. This value begins at -10 and must end at 10 while changing by exactly ± 1 at each step. Therefore, there must be a point at which $b - c = 1$ if the transformation is possible. We would have a reducible form, however, using $(x + c)(x + 1)$. Therefore, it is impossible to complete the process.

References

- [1] Edward J. Barbeau, *Polynomials*, Springer, New York, 1989.
- [2] E.R. Berlekamp, J.H. Conway, and R.K. Guy, *Winning Ways for your Mathematical Plays*, A.K. Peters, 2004.
- [3] Martin Gardner, *aha! Insight*, W.H. Freeman, San Francisco, 1978.
- [4] Richard Hoshino and John Grant McLoughlin, *Combinatorial Explorations*, Canadian Mathematical Society, Ottawa, 2005.
- [5] Ian Stewart, *Math Hysteria*, Oxford University Press, Oxford, 2004.