Problem of the Month

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This month, we will consider two problems with the same theme.

Problem 1 (2006 Gauss Contest (Grade 7)). A triangle can be formed having side lengths 4, 5, and 8. It is impossible however, to construct a triangle with side lengths 4, 5, and 10. Using the side lengths 2, 3, 5, 7, and 11, how many different triangles *with exactly two equal sides* can be formed?

So what is this all about? Just what are those mysterious first two sentences trying to tell us? Why can we make a triangle with certain side lengths and not with others? We could solve this problem in an intuitive way, but let's try to be more systematic.

With certain potential sets of side lengths, it makes more sense that a triangle cannot be formed than with other potential sets. For instance, it seems highly unlikely that we should be able to make a triangle with sides of length 1, 2, and 1000. With lengths that are closer together, it is not quite as clear (as in the case with sides of length 4, 5, and 10). What is the technical reason here?

This is an example of something called the *triangle inequality*, which says that in any triangle, the length of each side must be less than the sum of the lengths of the other two sides.

More technically, if $\triangle ABC$ has side lengths $AB = c$, $AC = b$, and $BC = a$, then we must have $c < a + b$, $b < a + c$, and $a < b + c$. How can we justify these facts? The easiest way is actually quite simple. Consider the two points A and B. What is the shortest path between A and B? Yes, you in the back. . . Yes, it is the straight line segment AB, whose length is c. Any other path from A to B is longer. In particular, going from A to B via C (a distance $AC + CB = b + a$) is longer, which means that $b + a > c$. We can obtain the inequalities $a + c > b$ and $b + c > a$ in a similar way.

At this stage, we could launch into the solution to Problem 1. However, let's hold off to make one more observation. Suppose that the lengths of the sides of $\triangle ABC$ satisfy $0 < a \leq b \leq c$. How many of the three inequalities $c < a + b$, $b < a + c$, and $a < b + c$ actually contain "useful" information? Since $b \geq a$ and $c > 0$, we get $b + c > a$ automatically. Similarly, $a + c > b$ automatically. Thus, only one of the three inequalities is worth considering, namely the one that says that the sum of the lengths of the two shorter sides is greater than the length of the longest side.

Solution to Problem 1: Consider a triangle with two equal sides. We write the side lengths as a, a, and b. Certainly $a + b > a$, which accounts for two of the three inequalities. The third states that $a + a > b$, or $2a > b$. In other words, if we are given the lengths a, a, and b, we need only check whether $b < 2a$ to determine whether the triangle inequality is satisfied by these lengths.

In this problem, the possible values for a and b are 2, 3, 5, 7, and 11. For each possible value of b, let's count the number of possible values for a with $2a > b$, remembering that a cannot equal b.

If $b = 2$, then a can be 3, 5, 7, or 11.
If $b = 3$, then a can be 2, 5, 7, or 11.
If $b = 5$, then a can be 3, 7, or 11. (a cannot be 2.)
If $b = 7$, then a can be 5 or 11. (a cannot be 2 or 3.)
If $b = 11$, then a can be 7. (a cannot be 2, 3, or 5.)

Adding up the possibilities, we discover that there are 14 different triangles that can be formed.

Wait! "Different" triangles? Yes, no two of them are congruent, since "side-side-side" is a valid check for congruency. "Can be formed"? Trickier, but still fine here—try to justify this on your own.

Here is a second problem which "sticks" to the same topic.

Problem 2 (2001 Gauss Contest (Grade 7)). A triangle can be formed having side lengths 4, 5, and 8. It is impossible, however, to construct a triangle with side lengths 4, 5, and 10. Ron has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. What is the shortest possible length of the longest of the eight sticks?

It looks like we have to use the pesky triangle inequality again. This problem also asks us to find the "minimum possible value of a maximum"—a standard type of problem, both in mathematics and in real life. It is like trying to determine the "worst case scenario".

Solution to Problem 2: Suppose that we write the lengths of the sticks in order as $a \leq b \leq c \leq d \leq \cdots$. Since a, b, and c do not form a triangle, we cannot have $c < a + b$; hence, $c \geq a + b$. To make c as small as possible given fixed a and b, we want $c = a + b$. Similarly, since b, c, and d cannot form a triangle, we want $d = b + c$, and so on.

Of course, to make everything as small as possible, we should start with a and b as small as possible. Since the smallest positive integer is 1, we choose $a = b = 1$. Using the idea of giving each new stick the sum of the lengths of the two previous sticks, we get lengths 1, 1, 2, 3, 5, 8, 13, and 21. This ensures that no triangle can be formed from any set of three sticks. Why? At each stage, the new longest stick will have a length equal to the sum of the two previous largest lengths and, therefore, at least as large as the sum of any two of the previous lengths.

Thus, the lengths are 1, 1, 2, 3, 5, 8, 13, and 21, and so the eighth stick has length 21. (Fibonacci strikes again!)

We see that the triangle inequality, though perhaps difficult to put one's finger on, can be useful. In fact, it appears throughout mathematics in many different guises—so always be on the lookout for it!