SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3059. [2005 : 334, 337] *Proposed by Gabriel Dospinescu, Paris, France.*

Let a, b, c, d be real numbers such that $a^2+b^2+c^2+d^2 \le 1$. Prove that

$$ab+bc+cd+da+ac+bd \;\le\; 4abcd+\frac{5}{4}\,.$$

I. *Solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA, modified by the editor.*

Note first that the AM–GM Inequality implies that

$$4\sqrt[4]{(abcd)^2} \;\le\; a^2+b^2+c^2+d^2 \;\le\; 1\,,$$

and hence, $\sqrt{|abcd|} \le \frac{1}{4}$, or

$$|abcd| \;\le\; \tfrac{1}{16}\,. \qquad (1)$$

Further, equality is attained in (1) if and only if $a^2=b^2=c^2=d^2$ and $a^2+b^2+c^2+d^2=1$; that is, if and only if $a^2=b^2=c^2=d^2=\frac{1}{4}$.

For real numbers x and y, the Cauchy–Schwarz Inequality implies that

$$x+y \;\le\; \sqrt{2}\sqrt{x^2+y^2}\,,$$

with equality if and only if $x=y\ge 0$. Using this and the AM–GM Inequality, we obtain

$$\begin{aligned}
(ab+cd)+(bc+da) &\le \sqrt{2}\sqrt{(ab+cd)^2+(bc+da)^2}\\
&= \sqrt{2}\sqrt{(a^2+c^2)\,(b^2+d^2)+4abcd}\\
&\le \sqrt{2}\sqrt{\left(\tfrac{1}{2}(a^2+c^2+b^2+d^2)\right)^2+4abcd}\\
&\le \sqrt{2}\sqrt{\tfrac{1}{4}+4abcd} \;=\; \tfrac{\sqrt{2}}{2}\sqrt{1+16abcd}\,,
\end{aligned}$$

with equality if and only if $ab+cd=bc+da\ge 0$ and $a^2+c^2=b^2+d^2=\frac{1}{2}$.

Similarly, we get

$$(bc+da)+(ac+bd) \;\le\; \tfrac{\sqrt{2}}{2}\sqrt{1+16abcd}\,,$$

with equality if and only if $bc+da=ac+bd\ge 0$ and $a^2+b^2=c^2+d^2=\frac{1}{2}$, and

$$(ab+cd)+(ac+bd) \;\le\; \tfrac{\sqrt{2}}{2}\sqrt{1+16abcd}\,,$$

with equality if and only if $ab+cd=ac+bd\ge 0$ and $a^2+d^2=b^2+c^2=\frac{1}{2}$.

Therefore,

$$\begin{aligned} ab + bc + cd + da + ac + bd & \\ &= \tfrac{1}{2}[(ab+cd) + (bc+da) + (bc+da) + (ac+bd) \\ &\qquad + (ab+cd) + (ac+bd)] \\ &\le \tfrac{3\sqrt{2}}{4}\sqrt{1+16abcd}\,, \end{aligned} \tag{2}$$

with equality if and only if $ab + cd = bc + da = ac + bd \ge 0$ and $a^2 = b^2 = c^2 = d^2 = \frac{1}{4}$.

Let $x = 1 + 16abcd$. From (1), we have $0 \le x \le 2$, and hence

$$(x+4)^2 - \left(3\sqrt{2}\sqrt{x}\right)^2 = x^2 - 10x + 16 = (x-8)(x-2) \ge 0\,,$$

with equality if and only if $x = 2$. Thus,

$$3\sqrt{2}\sqrt{1+16abcd} \le 16abcd + 5\,, \tag{3}$$

with equality if and only if $abcd = \frac{1}{16}$.

Finally, combining (1), (2), and (3), we obtain

$$ab + bc + cd + da + ac + bd \le \tfrac{3\sqrt{2}}{4}\sqrt{1+16abcd} \le 4abcd + \tfrac{5}{4}\,,$$

with equality if and only if $a^2 = b^2 = c^2 = d^2 = \frac{1}{4}$, $abcd = \frac{1}{16}$, and $ab+cd = bc+da = ac+bd \ge 0$; that is, if and only if $a = b = c = d = \pm\frac{1}{2}$.

II. *Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and the proposer.*

Let t be a real number, and let $S = ab + bc + cd + da + ac + bd$. Consider the polynomial

$$\begin{aligned} A(x) &= (x-a)(x-b)(x-c)(x-d) \\ &= x^4 - (a+b+c+d)x^3 + Sx^2 \\ &\qquad - (abc + bcd + abd + acd)x + abcd\,. \end{aligned}$$

Since $|p + iq| \ge |p|$, we obtain

$$\begin{aligned} |A(it)|^2 &= \Big|t^4 + it^3 \sum_{\text{cyclic}} a - St^2 - it \sum_{\text{cyclic}} abc + abcd\Big|^2 \\ &\ge \left|t^4 - St^2 + abcd\right|^2\,. \end{aligned}$$

On the other hand,

$$|A(it)|^2 = A(it)\cdot\overline{A(it)} = \prod_{\text{cyclic}}(a - it)\cdot\prod_{\text{cyclic}}(a+it) = \prod_{\text{cyclic}}(a^2+t^2)\,.$$

Thus,

$$\left|t^4 - St^2 + abcd\right|^2 \;\le\; \prod_{\text{cyclic}} (a^2 + t^2)\,.$$

Set $t = 1/2$ in this inequality, and then use the AM–GM Inequality and the given condition to obtain

$$\begin{aligned}\left|\tfrac{1}{16} - \tfrac{1}{4}S + abcd\right|^2 &\le \prod_{\text{cyclic}} \left(a^2 + \tfrac{1}{4}\right) \le \Big(\tfrac{1}{4}\sum_{\text{cyclic}}\left(a^2+\tfrac{1}{4}\right)\Big)^4 \\ &= \left(\tfrac{1}{4}(a^2+b^2+c^2+d^2+1)\right)^4 \le \tfrac{1}{16}\,.\end{aligned}$$

Therefore, $\left|\frac{1}{16} - \frac{1}{4}S + abcd\right| \le \frac{1}{4}$, which yields $S \le 4abcd + \frac{5}{4}$, completing the proof.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There were two incorrect (incomplete) solutions submitted.

3060. [2005 : 334, 337] *Proposed by Gabriel Dospinescu, Paris, France.*

Let a and b be positive real numbers such that $a < 2$. For each integer $n \ge 1$, let $x_n = \lfloor an + b \rfloor$. Prove that the sequence $\{x_n\}_{n\ge 1}$ has an infinite number of terms whose sum of digits is even. (Note: $\lfloor z \rfloor$ is the greatest integer not exceeding z.)

Solution by Tom Leong, Brooklyn, NY, USA, with some detail added by the editor.

If $0 < a \le 1$, then $\{x_n\}$ assumes every integer value greater than or equal to $\lfloor a+b \rfloor$, and the result is clearly true. Hence, we may assume that $1 < a < 2$. Then all the terms of the sequence $\{x_n\}$ are distinct. It is easy to verify (by induction or otherwise) that for $k \ge 2$, exactly half of the $9(10^{k-1})$ positive integers with k digits have an even sum of digits (and half have an odd sum).

The number of terms x_n with k digits is equal to the number of positive integers n such that $10^{k-1} \le an + b < 10^k$. We rewrite these inequalities as

$$\frac{10^{k-1} - b}{a} \;\le\; n \;<\; \frac{10^k - b}{a}\,.$$

If k is large enough so that $10^{k-1} > b$, then all integers n satisfying the above inequalities are positive. The number of such integers is at least

$$\frac{10^k - b}{a} - \frac{10^{k-1} - b}{a} - 1 \;=\; \frac{9\left(10^{k-1}\right)}{a} - 1\,.$$

Since $a < 2$, this number will be greater than $9\left(10^{k-1}\right)/2$ for all k sufficiently large (large enough so that $9\left(10^{k-1}\right) > \left(\frac{1}{a} - \frac{1}{2}\right)^{-1}$).

Thus, for all k sufficiently large, the number of terms x_n with k digits is greater than the number of k–digit positive integers with an odd sum of digits, and hence there exists a term x_n with k digits and an even sum of digits. The desired result follows.

Also solved by the proposer.

3061. [2005 : 335, 337] *Proposed by Gabriel Dospinescu, Paris, France.*

Find the smallest non-negative integer n for which there exists a non-constant function $f : \mathbb{Z} \to [0, \infty)$ such that for all integers x and y,

(a) $f(xy) = f(x)f(y)$, and

(b) $2f(x^2 + y^2) - f(x) - f(y) \in \{0, 1, \ldots, n\}$.

For this value of n, find all the functions f which satisfy (a) and (b).

Solution by Michel Bataille, Rouen, France, modified by the editor.

The solution makes use of the following known result.

Proposition 1. If p is prime, $p \equiv 3 \pmod 4$, and a and b are integers such that $p \mid (a^2 + b^2)$, then $p \mid a$ and $p \mid b$.

We show that the smallest n is $n = 1$.

Let $f : \mathbb{Z} \to [0, \infty)$ be a non-constant function satisfying (a). Since $f(1) = f(1 \cdot 1) = \big(f(1)\big)^2$, we have $f(1) = 1$ or $f(1) = 0$. But the latter yields $f(x) = f(x \cdot 1) = f(x)f(1) = 0$ for all $x \in \mathbb{Z}$, contradicting the fact that f is not constant. It follows that $f(1) = 1$. Similarly, $f(-1) = 1$ and $f(0) = 0$.

If $n = 0$, then f cannot satisfy (b), since $2f(x^2+y^2) \neq f(x)+f(y)$ for $x = 1$ and $y = 0$. The function $f_0 : \mathbb{Z} \to [0, \infty)$ defined by $f_0(0) = 0$ and $f_0(x) = 1$ for all non-zero integers x, is non-constant and clearly satisfies (a). Moreover, if $K_0(x, y) = 2f_0(x^2+y^2) - f_0(x) - f_0(y)$, we have $K(x, y) = 0$ if $x = y = 0$; otherwise $K_0(x, y) = 2 - f_0(x) - f_0(y)$ with at least one of $f_0(x)$, $f_0(y)$ equal to 1. In all cases, $K_0(x, y) \in \{0, 1\}$ and f_0 satisfies (b) as well.

For $n = 1$ we will show that in addition to f_0, the solutions for f are the functions $f_p : \mathbb{Z} \to [0, \infty)$ defined by $f_p(x) = 0$ if $p \mid x$ and $f_p(x) = 1$ otherwise, where p is prime and $p \equiv 3 \pmod 4$. Such a function f_p is not constant, and satisfies (a) (since $p \mid xy$ implies $p \mid x$ or $p \mid y$). Let $K_p(x, y) = 2f_p(x^2+y^2) - f_p(x) - f_p(y)$. If $p \mid x$ and $p \mid y$, then $p \mid x^2+y^2$ and $K_p(x, y) = 0$. If $p \mid x$, and $p \nmid y$, then p does not divide $x^2 + y^2$, and $K_p(x, y) = 1$. Finally, if $p \nmid x$ and $p \nmid y$, then $p \nmid x^2 + y^2$ by Proposition 1, and $K_p(x, y) = 0$. In all cases $K_p(x, y) \in \{0, 1\}$ and f_p satisfies (b); whence, f_p is a solution.

Conversely, let f be a solution. We will show that either $f = f_0$ or $f = f_p$ for some prime $p \equiv 3 \pmod 4$. First, we will show that $f(x)$ is either 0 or 1 for all $x \in \mathbb{Z}$. Let $x \in \mathbb{Z}$ be such that $f(x) \neq 0$. Condition (a) implies that $f(x^2) = \big(f(x)\big)^2$ and (b) yields

$$2\big(f(x)\big)^2 - f(x) \;=\; 2f(x^2+0) - f(x) - f(0) \in \{0,1\}\,.$$

It follows that $f(x)$ is equal to 0, 1, or $1/2$ for all integer x. But if $f(x) = 1/2$ and $f(x^2+1) \in \{0,\,1,\,1/2\}$, then (b) does not hold for $y = 1$.

Next, we will show that $f(2) = 1$ and $f(q) \neq 0$ for any prime q, $q \equiv 1 \pmod 4$. Since $f(1) = 1$, from (b) with $x = y = 1$ we conclude that $f(2) = 1$. If q is a prime, $q \equiv 1 \pmod 4$, then there are integers a and k such that $qk = a^2 + 1$ and

$$2f(k)f(q) \;=\; 2f(kq) \;=\; 2f(a^2+1^2) \;=\; f(a) + f(1) + \varepsilon\,,$$

where ε is equal to 0 or 1. Therefore, $2f(k)f(q) \geq f(1) = 1 \neq 0$ and $f(q) \neq 0$.

Suppose $f \neq f_0$. Then $f(x_0) = 0$ for some integer x_0 with $|x_0| > 1$. By condition (a), we must have $f(p) = 0$ for some prime factor p of x_0 and $p \equiv 3 \pmod 4$. No other prime $p' \equiv 3 \pmod 4$ can satisfy $f(p') = 0$. Otherwise, we would have $2f(p^2+p'^2) \in \{0,\,1\}$; whence $f(p^2+p'^2) = 0$. But then there is a prime factor p'' of $p^2+p'^2$ such that $f(p'') = 0$; thus, $p'' \equiv 3 \pmod 4$. By Proposition 1, this is a contradiction since p'' cannot be a factor of both p and p'. It follows that $f(x) = 0$ if and only if p is a factor of x and $f = f_p$.

Also solved by the proposer.

3062. [2005 : 335, 337] *Proposed by Gabriel Dospinescu, Paris, France.*

Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that

$$(ab+bc+ca)\left(\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a}\right) \;\geq\; \frac{3}{4}\,.$$

Essentially the same solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and the proposer.

Let x be any positive real number. In the Cauchy–Schwarz Inequality $|\overrightarrow{u}|^2\,|\overrightarrow{v}|^2 \geq \left(\overrightarrow{u}\cdot\overrightarrow{v}\right)^2$, we set

$$\overrightarrow{u} \;=\; \left(\frac{\sqrt{a}}{x+b}, \frac{\sqrt{b}}{x+c}, \frac{\sqrt{c}}{x+a}\right) \qquad \text{and} \qquad \overrightarrow{v} \;=\; \left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)\,.$$

Then, since $|\overrightarrow{v}| = a+b+c = 1$, we obtain

$$\frac{a}{(x+b)^2} + \frac{b}{(x+c)^2} + \frac{c}{(x+a)^2} \;\geq\; \left(\frac{a}{x+b} + \frac{b}{x+c} + \frac{c}{x+a}\right)^2\,.$$

Now use the Cauchy–Schwarz Inequality again, this time with

$$\begin{aligned} \overrightarrow{u} &= \left(\sqrt{\frac{a}{x+b}}, \sqrt{\frac{b}{x+c}}, \sqrt{\frac{c}{x+a}}\right) \\ \text{and} \quad \overrightarrow{v} &= \left(\sqrt{a(x+b)}, \sqrt{b(x+c)}, \sqrt{c(x+a)}\right) . \end{aligned}$$

Since $\overrightarrow{u} \cdot \overrightarrow{v} = a+b+c = 1$, we get

$$\begin{aligned} \left(\frac{a}{x+b} + \frac{b}{x+c} + \frac{c}{x+a}\right)^2 &\geq \frac{1}{\left(a(x+b)+b(x+c)+c(x+a)\right)^2} \\ &= \frac{1}{(x+ab+bc+ca)^2} . \end{aligned}$$

Then

$$\int_0^1 \left(\frac{a}{(x+b)^2} + \frac{b}{(x+c)^2} + \frac{c}{(x+a)^2}\right) dx \geq \int_0^1 \frac{dx}{(x+ab+bc+ca)^2} ;$$

that is,

$$\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a} \geq \frac{1}{(ab+bc+ca)(1+ab+bc+ca)} .$$

Using the well-known inequality $3(ab+bc+ca) \leq (a+b+c)^2$, we obtain $ab+bc+ca \leq \frac{1}{3}$. Therefore,

$$(ab+bc+ca)\left(\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a}\right) \geq \frac{1}{1+ab+bc+ca} \geq \frac{3}{4} ,$$

as desired. Equality holds if and only if $a = b = c = \frac{1}{3}$.

Also solved by ARKADY ALT, San Jose, CA, USA; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; SILOUANOS BRAZITIKOS, student, Trikala, Greece; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (two solutions); RONGZHENG JIAO, Yangzhou University, Yangzhou, China; VEDULA N. MURTY, Dover, PA, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Janous has also proven the following similar results. If a, b, and c are positive numbers with $a+b+c=1$, then

$$(ab+bc+ca)\left(\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1}\right) \leq \frac{3}{4} ,$$

and

$$(ab+bc+ca)\left(\frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{a^2+1}\right) \leq \frac{9}{10} .$$

3063. [2005 : 335, 337] *Proposed by Mohammed Aassila, Strasbourg, France.*

Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f\big(f(x)\big) + f(x) \;=\; 2x + a\,,$$

where a is a real constant.

Solution by Michel Bataille, Rouen, France, expanded by the editor.

It is readily verified that the function $f(x) = x + \frac{1}{3}a$ is a solution (for any a) and that, if $a = 0$, any function of the form $f(x) = -2x + c$ is a solution, where c is an arbitrary real constant. We will prove that these are the only solutions.

Let f be any function that satisfies the given conditions. Clearly, f is one-to-one, since $f(x_1) = f(x_2)$ implies $2x_1 + a = 2x_2 + a$, from which we get $x_1 = x_2$. Being also continuous, f must be strictly monotone on $\mathbb{R}$; hence, f must have a limit (finite or infinite) as $x \to \infty$ and as $x \to -\infty$. If $\lim\limits_{x\to\infty} f(x) = L \in \mathbb{R}$, then $\lim\limits_{x\to\infty} f\big(f(x)\big) + f(x) = f(L) + L$ (since f is continuous), which contradicts $\lim\limits_{x\to\infty}(2x+a) = \infty$. Thus, $\lim\limits_{x\to\infty} f(x) = \pm\infty$. Similarly, $\lim\limits_{x\to-\infty} f(x) = \pm\infty$. Since f is monotone, either $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$, or $\lim\limits_{x\to\infty} f(x) = -\infty$ and $\lim\limits_{x\to-\infty} f(x) = \infty$. Hence, f is a continuous bijection from $\mathbb{R}$ onto $\mathbb{R}$. Now we consider two cases separately:

Case (i). f is strictly increasing.

Let $x \in \mathbb{R}$ be arbitrary but fixed. Define the sequence $\{u_n\}$ as follows: $u_0 = x$, $u_1 = f(x)$, and $u_{n+1} = f(u_n)$ for all $n \geq 1$. Then the given functional equation implies that, for all $n = 0, 1, 2, \ldots$,

$$u_{n+2} + u_{n+1} \;=\; 2u_n + a\,. \tag{1}$$

We solve this recurrence relation by the usual method. The characteristic equation of the corresponding homogeneous relation is $t^2 + t - 2 = 0$, and the characteristic roots are 1 and -2. By inspection, a particular solution is $u_n = \frac{1}{3}na$. Hence, the general solution of (1) is $u_n = \alpha + \beta(-2)^n + \frac{1}{3}na$. Using $\alpha + \beta = u_0 = x$ and $\alpha - 2\beta + \frac{1}{3}a = u_1 = f(x)$, we easily find that $\alpha = \frac{1}{3}\left(2x + f(x) - \frac{1}{3}a\right)$ and $\beta = \frac{1}{3}\left(x - f(x) + \frac{1}{3}a\right)$. Hence,

$$u_n \;=\; \tfrac{1}{3}\left(2x + f(x) - \tfrac{1}{3}a\right) + \tfrac{1}{3}(-2)^n\left(x - f(x) + \tfrac{1}{3}a\right) + \tfrac{1}{3}na\,. \tag{2}$$

Since f is increasing, the sequence $\{u_n\}$ must be monotone; thus, the sign of $u_{n+1} - u_n$ must be fixed for $n = 0, 1, 2, \ldots$. The second term on the right side of (2) reveals that this is possible only if $x - f(x) + \frac{1}{3}a = 0$. Since this is true for all $x \in \mathbb{R}$, it follows that $f(x) = x + \frac{1}{3}a$.

Case (ii). f is strictly decreasing.

We first show that $a = 0$. Let $\phi(x) = f(x) - x$. Then ϕ is continuous and strictly decreasing. Since $\lim_{x\to-\infty} \phi(x) = \infty$ and $\lim_{x\to\infty} \phi(x) = -\infty$, we deduce that ϕ has exactly one real root. Thus, there exists a unique real number x_0 such that $f(x_0) = x_0$. From $2x_0 + a = f\bigl(f(x_0)\bigr) + f(x_0) = 2x_0$, we then have $a = 0$.

Let $g = f^{-1}$ denote the inverse of f. For an arbitrary but fixed real number x, define the sequence $\{v_n\}$ as follows: $v_0 = f(x)$, $v_1 = x$, and $v_{n+1} = g(v_n)$ for all $n \geq 1$. Since $f \circ f$ is also a bijection, there exists $y \in \mathbb{R}$ such that $f\bigl(f(y)\bigr) = x$. Then the functional equation $f\bigl(f(y)\bigr) + f(y) = 2y$ becomes $x + g(x) = 2g\bigl(g(x)\bigr)$, which implies that, for all $n = 0, 1, 2, \ldots$,

$$2v_{n+2} = 2g(v_{n+1}) = 2g\bigl(g(v_n)\bigr) = g(v_n) + v_n = v_{n+1} + v_n .$$

Solving this recurrence relation as in Case (i), we find that

$$v_n = \tfrac{1}{3}\bigl(f(x) + 2x\bigr) + \tfrac{2}{3}\left(-\tfrac{1}{2}\right)^n \bigl(f(x) - x\bigr) .$$

Letting $m = \frac{1}{3}\bigl(f(x) + 2x\bigr)$, we have $\lim_{n\to\infty} v_n = m$. From the recurrence relation $v_{n+1} = g(v_n)$, we obtain $g(m) = m$ and hence, $f(m) = m$. Since x_0 is the only fixed point of f, we infer that $x_0 = m = \frac{1}{3}\bigl(f(x) + 2x\bigr)$. Since this is true for all $x \in \mathbb{R}$, we conclude that $f(x) = -2x + 3x_0 = -2x + c$, where $c = 3x_0$. Our proof is complete.

Also solved by ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer. There were also an incomplete solution and a solution which gave the correct answers but with faulty argument.

3064. [2005 : 397, 399] *Proposed by J. Chris Fisher, University of Regina, Regina, SK.*

(a) Starting with four points A, B, C, D in the plane, no three of which are collinear, let P, Q, R, S be the mid-points of AB, CD, AC, BD, respectively. Let L be the point of intersection of AQ and DP, and let M be the point of intersection of BR and CS. Prove that the mid-point of BC lies on the line LM if and only if $AD \parallel BC$.

(b) Let A_0, A_1, A_2, A_3, and A_4 be the vertices of a non-degenerate pentagon. Define a *median* to be a line that joins a vertex A_j either to the mid-point of the opposite side $A_{j+2}A_{j-2}$ or to the mid-point of the opposite diagonal $A_{j+1}A_{j-1}$ (where subscripts are taken modulo 5). Prove that the pentagon is affinely regular if and only if the ten medians are concurrent.

The result is based on a theorem of Zvonco Čerin, *Journal of Geometry*, 77 (2003), 22–34.

Note: A pentagon is said to be *affinely regular* if it is the image under a linear transformation of a regular pentagon or a regular pentagram.

(a) *Solution by Michel Bataille, Rouen, France.*

We shall see that the claim in (a) is not correct: although $AD \parallel BC$ does imply that the mid-point of BC lies on LM, the converse is false. We choose affine coordinates with the origin at B such that $A(0,2)$, $C(2,0)$, and $D(2a,2b)$ for real numbers a, b. Then $P(0,1)$, $Q(1+a,b)$, $R(1,1)$, $S(a,b)$, and the equations of the required lines are:

$$\begin{aligned} AQ: &\quad (b-2)x-(a+1)y+2(a+1) \;=\; 0\,,\\ DP: &\quad (2b-1)x-2ay+2a \;=\; 0\,,\\ BR: &\quad x-y \;=\; 0\,,\\ CS: &\quad bx-(a-2)y-2b \;=\; 0\,. \end{aligned}$$

From the equations of BR and CS, we find that $M\left(\dfrac{2b}{b-a+2}, \dfrac{2b}{b-a+2}\right)$. Denoting by I the mid-point of BC, we have $I(1,0)$ and IM has equation $2bx-(b+a-2)y-2b=0$. It follows that I lies on the line LM if and only if L lies on the line IM; that is, the lines AQ, DP, IM pass through a point:

$$\begin{vmatrix} b-2 & -a-1 & 2a+2 \\ 2b-1 & -2a & 2a \\ 2b & 2-a-b & -2b \end{vmatrix} \;=\; 0\,,$$

or finally,

$$(1-b)(a^2-ab-3a-4b+2) \;=\; 0\,. \tag{1}$$

Observe that $AD \parallel BC$ is equivalent to $b=1$. Thus, if $AD \parallel BC$, then the relation (1) is satisfied and I lies on LM, as desired. But not conversely—we may have I on LM but AD *not* parallel to BC. For example, this is the case if $a=-2$ and $b=6$ in the calculation above. It is readily checked that $L\left(\frac{4}{5},-\frac{6}{5}\right)$ and $M\left(\frac{6}{5},\frac{6}{5}\right)$, so that I lies on LM (indeed, I is the mid-point of LM), but AD is not parallel to BC (AD is in the direction of the vector $(-4,10)$, while BC has the direction vector $(2,0)$).

(b) *Incomplete solutions by Bataille and the proposer, completed by the editor.*

Any median of a regular pentagon or pentagram is a line of symmetry: the line from a vertex to the midpoint of the opposite side also passes through the midpoint of the opposite diagonal, so a regular pentagon has just five medians, and all five pass through the centre of the circumcircle. Affine transformations preserve mid-points and concurrency, so the five medians of affinely regular pentagons and pentagrams are concurrent.

For the converse, we let any four of the vertices of our pentagon, labeled consecutively, play the role of $ABCD$ from part (a). The concurrence of the medians says that $L=M$. We shall see that this is sufficient to prove that $AD \parallel BC$ and either $AD : BC = \tau : 1$ or $BC : AD = \tau : 1$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden section, which is the ratio of the diagonal of a regular pentagon to a side. The condition holds for each set of four points;

thus, we can uniquely construct vertices A_3 and A_4 of an affinely regular pentagon from any given triangle A_0, A_1, A_2. It therefore remains to show, in the notation of part (a), that $b = 1$ and a is either τ or its reciprocal. To satisfy the condition that $L = M$, we must have both AQ and DP meeting $BR : x - y = 0$ at M. Setting $y = x$ in the equation for DP (in part (a)), we find that

$$x = \frac{2a}{-2b+1+2a} = \frac{2b}{b-a+2},$$

where the last entry is the x–coordinate of M. The same process with AQ yields

$$x = \frac{2(1+a)}{3-b+a} = \frac{2b}{b-a+2}.$$

These equations represent a pair of conics in the variables a and b,

$$\begin{aligned} a^2 - 2b^2 + ab - 2a + b &= 0, \\ \text{and} \quad a^2 - b^2 - a + 2b - 2 &= 0. \end{aligned}$$

Setting $b = 1$, both these equations reduce to $a^2 - a - 1 = 0$, whose solution is $a = \tau$ or $a = -\frac{1}{\tau}$.

The common points of these two conics are, therefore, $(a,b) = (\tau, 1)$, $\left(-\frac{1}{\tau}, 1\right)$, $(2, 0)$, and the point at infinity of the line $a = b$. Of these, only $a = 2$ and $b = 0$ produces a point of the affine plane that is a zero of the second factor $a^2 - ab - 3a - 4b + 2$ in the determinant of part (a); but these values force the points $B(0,0)$, $C(2,0)$, and $D(2a, 2b)$ to be collinear, contrary to the assumption that we start with five non-collinear points. The other two common points have $b = 1$ as desired; the choice $(a,b) = (\tau, 1)$ produces an affinely regular pentagon while $(a,b) = \left(-\frac{1}{\tau}, 1\right)$ produces an affinely regular pentagram.

Part (a) *also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria* (*including a counterexample*)*; part* (b) *solved by JOEL SCHLOSBERG, Bayside, NY, USA.*

3065. [2005 : 397, 399] *Proposed by Gabriel Dospinescu, Paris, France.*

Let ABC be an acute-angled triangle, and let M be an interior point of the triangle. Prove that

$$\frac{1}{MA} + \frac{1}{MB} + \frac{1}{MC} \geq 2\left(\frac{\sin\angle AMB}{AB} + \frac{\sin\angle BMC}{BC} + \frac{\sin\angle CMA}{CA}\right).$$

Solution by Michel Bataille, Rouen, France.

Let d_a, d_b, and d_c denote the distances from point M to the sides BC, CA, and AB, respectively. Using the Law of Sines for triangle BMC, we obtain

$$2 \cdot \frac{\sin \angle BMC}{BC} = \frac{\sin \angle MCB}{MB} + \frac{\sin \angle MBC}{MC}$$
$$= \frac{d_a}{MB \cdot MC} + \frac{d_a}{MB \cdot MC} = 2 \cdot \frac{d_a}{MB \cdot MC}$$

Thus,

$$\frac{\sin \angle BMC}{BC} = \frac{d_a}{MB \cdot MC}.$$

Similarly,

$$\frac{\sin \angle CMA}{CA} = \frac{d_b}{MC \cdot MA} \quad \text{and} \quad \frac{\sin \angle AMB}{AB} = \frac{d_c}{MA \cdot MB}.$$

It follows that the proposed inequality is equivalent to the inequality

$$MB \cdot MC + MC \cdot MA + MA \cdot MB \geq 2(MA \cdot d_a + MB \cdot d_b + MC \cdot d_c),$$

which is known to be true (see [1] or [2]). This completes the proof.

References

[1] M. Bataille, Solution to problem 85G, *Mathematical Gazette*, 86(2002), pp. 149–151.

[2] A. Oppenheim, The Erdős Inequality and Other Inequalities for a Triangle, *American Mathematical Monthly*, 68(1961), pp. 226–230.

Also solved by ARKADY ALT, San Jose, CA, USA; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3066. [2005 : 397, 400] *Proposed by Gabriel Dospinescu, Paris, France.*

Given an integer $n > 2$, let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be subsets of $S = \{1, 2, \ldots, n\}$ with the property that for all $i, j \in S$, the subsets A_i and B_j have exactly one element in common. Prove that, if there are at least two distinct subsets among $B_1, B_2, \ldots, B_n$, then there exists a non-empty subset $T \subseteq S$ that has an even number of elements in common with each of the subsets $A_1, A_2, \ldots, A_n$.

Solution by Tom Leong, Brooklyn, NY, USA.

Let x_{ij} denote the element common to A_i and B_j, and let $B_1 \neq B_2$, say. Then $T = (B_1 \cup B_2) \setminus (B_1 \cap B_2)$ is non-empty and meets each of $A_1, A_2, \ldots, A_n$ in zero or two elements. Indeed, if $x_{i1} \in B_1 \cap B_2$, then $x_{i1} = x_{i2}$ and $T \cap A_i$ is empty; while if $x_{i1} \in B_1 \setminus B_2$, then $x_{i2} \in B_2 \setminus B_1$ and $T \cap A_i = \{x_{i1}, x_{i2}\}$.

Also solved by KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

As in the featured solution, most solvers used the symmetric difference $B_1 \Delta B_2$ (which is the set of all elements contained in exactly one of the two sets B_1 and B_2) for the desired subset T. Only Schlosberg observed that the symmetric difference has an even intersection with *each A_i more generally when each A_i has an odd intersection with each B_j (rather than just a single common element). Specifically,*

> Let A_1, A_2, ..., A_m, B, and C be subsets of a finite set such if $|A_i \cap B|$ and $|A_i \cap C|$ are both odd for all i, then $|A_i \cap (B \Delta C)|$ is even for all i.

3067. [2005 : 398, 400] *Proposed by Gabriel Dospinescu, Paris, France.*

Find all functions $f : (0,\infty) \to (0,\infty)$ such that

1. $f(f(f(x))) + 2x = f(3x)$ for all $x > 0$, and
2. $\lim\limits_{x\to\infty} \big(f(x) - x\big) = 0$.

Composite of very similar solutions by Joel Schlosberg, Bayside, NY, USA; and the proposer.

The function $f(x) = x$ clearly has the required properties. We will prove that it is the only function with these properties.

Suppose that a function $f : (0,\infty) \to (0,\infty)$ has properties 1 and 2. For any $x > 0$, property 1 implies that

$$f(x) \;=\; \tfrac{2}{3}x \;+\; f\Big(f\big(f\big(\tfrac{1}{3}x\big)\big)\Big) \;>\; \tfrac{2}{3}x\,.$$

Define a sequence $\{a_n\}_{n=1}^{\infty}$ by setting $a_1 = \frac{2}{3}$ and $a_{n+1} = \frac{1}{3}a_n^3 + \frac{2}{3}$ for all $n \in \mathbb{N}$. We will prove by induction that $f(x) > a_n x$ for all x and n.

The case $n = 1$ was proven above. Assume that $f(x) > a_n x$ for some $n \in \mathbb{N}$ and all $x > 0$. Then, for all $x > 0$,

$$f\Big(f\big(f\big(\tfrac{1}{3}x\big)\big)\Big) \;>\; a_n f\big(f\big(\tfrac{1}{3}x\big)\big) \;>\; a_n^2 f\big(\tfrac{1}{3}x\big) \;>\; a_n^3 \cdot \tfrac{1}{3}x\,,$$

and hence,

$$f(x) \;=\; \tfrac{2}{3}x \;+\; f\Big(f\big(f\big(\tfrac{1}{3}x\big)\big)\Big) \;>\; \tfrac{2}{3}x + a_n^3 \cdot \tfrac{1}{3}x \;=\; a_{n+1}x\,.$$

This completes the induction.

Applying the AM–GM Inequality, we get $a_{n+1} = \dfrac{a_n^3 + 1^3 + 1^3}{3} \geq a_n$ for all n. Thus, the sequence $\{a_n\}$ is increasing. Furthermore, $0 < a_n < 1$ for all n, as can be shown by an easy induction. Therefore, the sequence converges. Letting u denote its limit, we have

$$u \;=\; \lim_{n\to\infty} a_{n+1} \;=\; \lim_{n\to\infty} \big(\tfrac{1}{3}a_n^3 + \tfrac{2}{3}\big) \;=\; \tfrac{1}{3}u^3 + \tfrac{2}{3}\,.$$

Hence, $u^3 - 3u + 2 = 0$, from which we get $u = 1$ or $u = -2$. But $u = -2$ is impossible, since $a_n > 0$ for all n. Therefore $u = 1$. Since $f(x) > a_n x$ for all n and x, we obtain, for all $x > 0$,

$$f(x) \;\geq\; \lim_{n\to\infty} a_n x \;=\; x\,.$$

For all $x > 0$, since $f(x) \geq x$, we have

$$f\Big(f\big(f(x)\big)\Big) \;\geq\; f\big(f(x)\big) \;\geq\; f(x)\,,$$

and hence, using property (a) again,

$$f(3x) - 3x \;=\; f\Big(f\big(f(x)\big)\Big) - x \;\geq\; f(x) - x\,.$$

By induction, we then obtain $f(x) - x \leq f(3^n x) - 3^n x$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty}\big(f(3^n x) - 3^n x\big) = 0$ (by property 2), we have $f(x) - x \leq 0$; that is, $f(x) \leq x$.

We have shown that $x \leq f(x) \leq x$ for all $x > 0$. We conclude that $f(x) = x$ for all $x > 0$.

There was one incomplete solution submitted.

3068. [2005 : 398, 400] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\sqrt{1+\frac{48a}{b+c}}+\sqrt{1+\frac{48b}{c+a}}+\sqrt{1+\frac{48c}{a+b}} \;\geq\; 15\,,$$

and determine when there is equality.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Without loss of generality, we may assume that $a \geq b \geq c \geq 0$ and that $a+b+c = 1$. Let $f(x) = \sqrt{1+\dfrac{48x}{1-x}}$ for $0 \leq x < 1$. Then

$$\begin{aligned} f'(x) &= \frac{24}{\sqrt{(1-x)^3(1+47x)}} > 0 \\ \text{and}\quad f''(x) &= \frac{48(47x-11)}{\sqrt{(1-x)^5(1+47x)^3}}\,. \end{aligned}$$

The tangent line to the graph of $f(x)$ at the point $\left(\frac{1}{3}, 5\right)$ has equation $T(x) = \dfrac{54x+7}{5}$. Setting $f(x) = T(x)$, we obtain $12(3x-1)^2(27x-2) = 0$, from which we see that the graphs of the functions f and T intersect again at $x = \frac{2}{27}$. Define

$$g(x) \;=\; \begin{cases} T(x) & \text{if } \frac{2}{27} \leq x < \frac{1}{3}, \\ f(x) & \text{if } \frac{1}{3} \leq x < 1. \end{cases}$$

Clearly, the function g is convex and $g(x) \leq f(x)$ for $\frac{2}{27} \leq x < 1$.

If $b \le \frac{2}{27}$, then $a = 1 - b - c \ge \frac{23}{27}$, and therefore,

$$f(a) \ \ge\ f\left(\tfrac{23}{27}\right) \ =\ \sqrt{277} \ >\ 15\,,$$

which implies that the original inequality holds. Hence, we can further assume that $b > \frac{2}{27}$.

If $c > \frac{2}{27}$, then, applying Jensen's Inequality, we obtain

$$\begin{aligned} f(a)+f(b)+f(c) &\ \ge\ g(a)+g(b)+g(c) \\ &\ \ge\ 3g\left(\tfrac{1}{3}(a+b+c)\right) \ =\ 3g\left(\tfrac{1}{3}\right) \ =\ 15\,, \end{aligned}$$

with equality if and only if $a = b = c = \frac{1}{3}$.

If $\frac{1}{17} < c \le \frac{2}{27}$, then $f(c) > f(\frac{1}{17}) = 2$ and, applying Jensen's Inequality, we obtain

$$f(a)+f(b) \ \ge\ g(a)+g(b) \ \ge\ 2g\left(\tfrac{1}{2}(a+b)\right) \ \ge\ 2g\left(\tfrac{25}{54}\right) \ >\ 13\,.$$

Thus, $f(a) + f(b) + f(c) > 15$, and the original inequality holds again.

Finally, consider $c \le \frac{1}{17}$. Then, applying Jensen's Inequality, we have

$$\begin{aligned} f(a)+f(b)+f(c) &\ \ge\ g(a)+g(b)+f(c) \ \ge\ 2g\left(\tfrac{1}{2}(a+b)\right)+f(c) \\ &\ =\ 2f\left(\tfrac{1}{2}(a+b)\right)+f(c) \ =\ 2f\left(\tfrac{1}{2}(1-c)\right)+f(c)\,. \end{aligned}$$

Define

$$h(x) \ =\ 2f\left(\frac{1-x}{2}\right)+f(x) \ =\ 2\sqrt{\frac{49-47x}{1+x}}+\sqrt{\frac{1+47x}{1-x}}$$

for $0 \le x \le \frac{1}{17}$. Then $h(0) = 15$ and

$$h'(x) \ =\ 24\left(\frac{1}{\sqrt{(1-x)^3(1+47x)}} - \frac{4}{\sqrt{(1+x)^3(49-47x)}}\right)\,.$$

Now, $(1+x)^3(49-47x) - 16(1-x)^3(1+47x) = (3x-1)k(x)$, where $k(x) = 235x^3 - 699x^2 + 505x - 33$. It is easy to verify that $k(\frac{1}{17}) < 0$, $k(1) > 0$, $k(\frac{3}{2}) < 0$, and $k(2) > 0$. Hence, $k(x) < 0$ for $0 \le x \le \frac{1}{17}$. Thus, $h'(x) > 0$, and therefore, $h(x) \ge h(0) = 15$ for $0 \le x \le \frac{1}{17}$.

This completes the proof. In summary, equality holds if and only if $a = b = c$ or two of a, b, c are equal while the third is 0.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

The proposer has proven the following, slightly more general result: If $a \ge 0$, $b \ge 0$, $c \ge 0$, and $0 < m \le 24$, then

$$\sqrt{1+\frac{2ma}{b+c}}+\sqrt{1+\frac{2mb}{c+a}}+\sqrt{1+\frac{2mc}{a+b}} \ \ge\ 3\sqrt{1+m}\,.$$

3069. [2005 : 398, 400] *Proposed by Cristinel Mortici, Valahia University of Targoviste, Romania.*

Let A, $B \in M_2(\mathbb{C})$ be such that $(AB)^2 = A^2B^2$. Prove that

$$\det(I + AB - BA) \ = \ 1\,.$$

Solution by Michel Bataille, Rouen, France.

We will denote by $\operatorname{tr}(X)$ the trace of $X \in M_2(\mathbb{C})$. Let $S = AB - BA$. Since $\operatorname{tr}(MN) = \operatorname{tr}(NM)$ for any pair of $n \times n$ matrices, $\operatorname{tr}(S) = 0$ and, moreover, $\operatorname{tr}(S^2) = 0$, since

$$\begin{aligned}\operatorname{tr}(S^2) &= \operatorname{tr}\big((AB)^2 - AB^2A - BA^2B + (BA)^2\big)\\ &= \operatorname{tr}\big((AB)^2\big) - \operatorname{tr}\big((AB^2)A\big) - \operatorname{tr}\big(B(A^2B)\big) + \operatorname{tr}\big(B(ABA)\big)\\ &= \operatorname{tr}\big((AB)^2\big) - \operatorname{tr}\big(A^2B^2\big) - \operatorname{tr}\big(A^2B^2\big) + \operatorname{tr}\big((AB)^2\big) \ = \ 0\,,\end{aligned}$$

where the last equality results from the hypothesis $(AB)^2 = A^2B^2$.

By the Cayley-Hamilton Theorem, $S^2 - \big(\operatorname{tr}(S)\big)S + \det(S)I = 0$; that is, $S^2 = -\det(S)I$. Taking traces gives $0 = \operatorname{tr}(S^2) = -2\det(S)$. Thus, $\operatorname{rank}(S) < 2$ and $S^2 = 0$. [*Editor's comment:* Those who prefer to avoid the Cayley-Hamilton Theorem can observe that $\operatorname{tr}(S) = 0$ implies that the eigenvalues of S are $\pm\lambda$, while $\operatorname{tr}(S^2) = 0$ implies that $2\lambda^2 = 0$; therefore, $\lambda = 0$ and S is nilpotent.]

Should $S = 0$, then $\det(I+S) = \det(I) = 1$. Otherwise, $\operatorname{rank}(S) = 1$ and S is similar to $T = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ for some non-zero complex number α. (If u is a non-zero vector in $\operatorname{range}(S)$ and v is such that $\{\mathrm{u}, \mathrm{v}\}$ is a basis of $\mathbb{C}^2$, we have $S\mathrm{u} = 0$ (since $S^2 = 0$) and $S\mathrm{v} = \alpha\mathrm{u}$ for some complex α.) Therefore, $I + S$ is similar to $I + T = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, and $\det(I + S) = \det(I + T) = 1$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RONGZHENG JIAO, Yangzhou University, Yangzhou, China; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3070. [2005 : 398, 400] *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let x_1, x_2, ..., x_n be positive real numbers such that

$$x_1 + x_2 + \cdots + x_n \ \geq \ x_1x_2\cdots x_n\,.$$

Prove that

$$(x_1x_2\cdots x_n)^{-1}\left(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}\right) \ \geq \ \sqrt[n-1]{n^{n-2}}\,,$$

and determine when there is equality.

Solution by Joel Schlosberg, Bayside, NY, USA.

For $n = 2$, we are required to prove that $(x_1x_2)^{-1}(x_1 + x_2) \geq 1$ if $x_1 + x_2 \geq x_1x_2$. This is trivially true. Equality holds in this case if and only if $x_1 + x_2 = x_1x_2$.

Suppose now that $n \geq 3$. By the AM–GM Inequality,

$$\begin{aligned} \frac{x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}}{n} &\geq \left(x_1^{n-1}x_2^{n-1}\cdots x_n^{n-1}\right)^{\frac{1}{n}} \\ &= (x_1x_2\cdots x_n)^{\frac{n-1}{n}} . \end{aligned}$$

Thus,

$$(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1})^{\frac{n(n-2)}{n-1}} \geq n^{\frac{n(n-2)}{n-1}}(x_1\cdots x_n)^{n-2} , \qquad (1)$$

with equality if and only if $x_1^{n-1} = x_2^{n-1} = \cdots = x_n^{n-1}$, which is equivalent to $x_1 = x_2 = \cdots = x_n$.

Since $n - 1 > 1$, the Power Mean Inequality gives us

$$\left(\frac{x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}}{n}\right)^{\frac{1}{n-1}} \geq \frac{x_1 + x_2 + \cdots + x_n}{n} ;$$

that is,

$$(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1})^{\frac{1}{n-1}} \geq n^{-\frac{n-2}{n-1}}(x_1 + x_2 + \cdots + x_n) . \qquad (2)$$

Equality holds here if and only if $x_1 = x_2 = \cdots = x_n$.

Multiplying (1) and (2), we get

$$(x_1^{n-1} + \cdots + x_n^{n-1})^{n-1} \geq n^{n-2}(x_1\cdots x_n)^{n-2}(x_1 + \cdots + x_n) .$$

If $x_1 + x_2 + \cdots + x_n \geq x_1\cdots x_n$, then we conclude that

$$(x_1^{n-1} + \cdots + x_n^{n-1})^{n-1} \geq n^{n-2}(x_1\cdots x_n)^{n-1} .$$

and the required inequality follows. Furthermore, we have equality if and only if $x_1 = x_2 = \cdots = x_n$ and $x_1 + x_2 + \cdots + x_n = x_1x_2\cdots x_n$, which occurs if and only if the common value x satisfies $nx = x^n$. Therefore, we get equality if and only if $x_1 = \cdots = x_n = n^{\frac{1}{n-1}}$.

Also solved by MICHEL BATAILLE, Rouen, France; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. The inequality only was proved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; and YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON.

3071. [2005 : 398, 400] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $k > -1$ be a fixed real number. Let a, b, and c be non-negative real numbers such that $a+b+c=1$ and $ab+bc+ca>0$. Find

$$\min\left\{\frac{(1+ka)(1+kb)(1+kc)}{(1-a)(1-b)(1-c)}\right\}.$$

Solution by Michel Bataille, Rouen, France, modified by the editor.

The required minimum is $\min\left\{\frac{1}{8}(k+3)^3,(k+2)^2\right\}$.

First, we establish the following inequality:

$$4(ab+bc+ca) \;\le\; 1+9abc\,. \tag{1}$$

Using the fact that $a+b+c=1$, we get

$$\begin{aligned}
&1-4(ab+bc+ca)+9abc\\
&\qquad = \;(a+b+c)^3-4(a+b+c)(ab+bc+ca)+9abc\\
&\qquad = \;a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) \;\ge 0\,,
\end{aligned}$$

where the last line is Schur's Inequality. This proves (1).

We also claim that

$$ab+bc+ca \;\ge\; 9abc\,. \tag{2}$$

Indeed,

$$\begin{aligned}
ab+bc+ca &= (ab+bc+ca)(a+b+c)\\
&= a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3abc\\
&\ge 6abc+3abc \;=\; 9abc\,,
\end{aligned}$$

where the inequality follows by an application of the AM–GM Inequality.

Turning back to the problem, we note that it is not possible for a, b, or c to equal 1. If $a=1$, for example, then $b=c=0$, which means that $ab+bc+ca=0$, a contradiction. Thus, a, b, $c\in[0,1)$. Let

$$\begin{aligned}
Q(a,b,c) &= \frac{(1+ka)(1+kb)(1+kc)}{(1-a)(1-b)(1-c)}\\
&= \frac{k^3abc+k^2(ab+bc+ca)+k+1}{ab+bc+ca-abc}\\
&= k^2+(k+1)\frac{k^2abc+1}{ab+bc+ca-abc}\,.
\end{aligned}$$

Note that $Q\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=\frac{1}{8}(k+3)^3$ and $Q\left(0,\frac{1}{2},\frac{1}{2}\right)=(k+2)^2$.

Case 1. $k^2\le 5$.

We prove that $Q(a,b,c)\ge Q\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. Since $k+1>0$, a straightforward calculation shows that this inequality is equivalent to

$$k^2(ab+bc+ca-9abc)+27(ab+bc+ca-abc) \;\le\; 8\,. \tag{3}$$

The term involving k^2 is non-negative, in view of (2). Since $k^2 \le 5$, the left side of (3) is at most $8\big(4(ab+bc+ca) - 9abc\big)$ and (3) follows from (1). Thus, $Q\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) = \frac{1}{8}(k+3)^3$ is the minimum value of Q.

Case 2. $k^2 \ge 5$.

We prove that $Q(a,b,c) \ge Q\left(0,\frac{1}{2},\frac{1}{2}\right)$. Since $k+1>0$, we find that this inequality is equivalent to

$$1 + 4\big(abc - (ab+bc+ca)\big) + k^2abc \;\ge\; 0\,.$$

This holds by (1) because $k^2 \ge 5$. Thus, $Q\left(0,\frac{1}{2},\frac{1}{2}\right) = (k+2)^2$ is the minimum value of Q.

Noticing that

$$\tfrac{1}{8}(k+3)^3 - (k+2)^2 \;=\; \tfrac{1}{8}(k+1)(k^2-5)\,,$$

we see that $\frac{1}{8}(k+3)^3 \ge (k+2)^2$ if $k^2 \ge 5$ and $(k+2)^2 \ge \frac{1}{8}(k+3)^3$ if $k^2 \le 5$. The announced result follows.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; RONGZHENG JIAO, Yangzhou University, Yangzhou, China; JOEL SCHLOSBERG, Bayside, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.