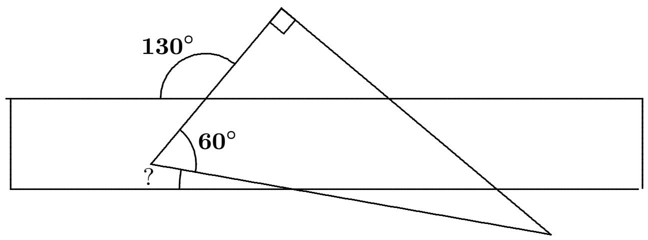
Mayhem Solutions

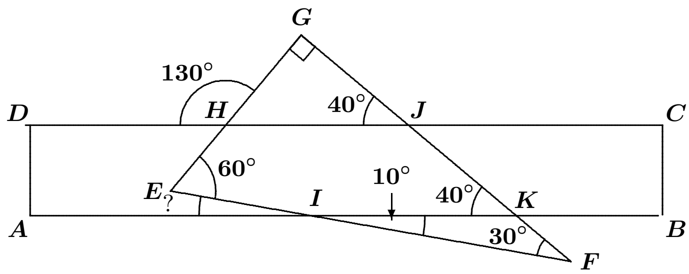
M201. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

A student drops his $30°$–$60°$–$90°$ triangle on his ruler so that a $130°$ angle appears as in the diagram below. What is the measure of the other marked angle?



Solution by Glenier L. Bello-Burguet, student, 3^{ro} de ESO Instituto Hermanos D'Elhuyar, Logroño, Spain.

We label the diagram as shown. The angle we seek is $\angle AIE = 10°$.



We are given $\angle GHD = 130°$ and also the angles of triangle EFG, namely $\angle GEF = 60°$, $\angle FGE = 90°$, $\angle EFG = 30°$. Thus, $\angle HJG = 40°$ (since $\angle HJG + \angle HGJ = \angle GHD$), and then the corresponding $\angle JKI$ also equals $40°$. Hence, $\angle KIF = 10°$ (because $\angle KIF + \angle EFG = \angle JKI$) and, therefore, also $\angle AIE = 10°$.

Also solved by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

M202. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

(a) By the end of the season, Samuel had made more than 50% of his foul shots, even though at the start of the season his average was below 50%. Show that there was a time during the season when his average was exactly 50%.

(b) For what other percentages p can you be certain that the average was exactly p at some time when you know only that the average was below p and then above p at a later time?

Solution by the Mayhem Staff.

(a) For the purpose of contradiction, assume that there was no time during the season when Samuel's average was exactly 50%. Then there must have been a time when, by sinking a foul shot, he increased his average from a fraction $a/b < 1/2$ to $(a+1)/(b+1) > 1/2$, where a and b are integers with $a \geq 0$ and $b > 0$. Then $2a < b$ and $2a + 2 > b + 1$. Together, these inequalities imply that $2a + 1 < b + 1 < 2a + 2$, which is impossible, since the integer $b + 1$ cannot lie strictly between two consecutive integers. This contradiction establishes the existence of a point in the season when Samuel had an average of 50%.

(b) By generalizing the argument in part (a) (using a fraction c/d in place of $1/2$), we can show that the percentages we seek are those of the form $c/(c+1)$, where c is a positive integer.

[*Ed:* This problem was explored in the Problem of the Month feature in the April 2006 issue of ***Crux Mathematicorum*** [2], and there is even a Proof Without Words treatment which has recently appeared [1].]

References

[1] Robert J. MacG. Dawson, Putnam Proof Without Words, *Mathematics Magazine*, 79 (2006), p. 149.

[2] Ian VanderBurgh, Problem of the Month, *Crux Mathematicorum with Mathematical Mayhem*, 32 (2006), pp. 147–148.

M203. *Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

Chuck goes into the local 7-11 store and buys four items. The bill totals $\$7.11$. He notices that the product of the four prices is exactly 7.11. What are the prices of the four items?

[*Ed:* The proposer has indicated that the problem does not originate with him, nor does he know its origin.]

Solution by the University of Regina Problem Group.

The items cost $\$1.20$, $\$1.25$, $\$1.50$, and $\$3.16$. We found the problem in *How To Solve It: Modern Heuristics*, by Zbigniew Michalewicz and David B. Fogel (Springer, 1998), Puzzle III, pages 49–53. The problem seems not to have originated with these authors, because they refer to it as the "so-called '7–11' problem", but they do not provide a reference. Our approach was essentially the same as theirs: convert the prices into cents (so that we can use integer arithmetic), systematically list all possible combinations of feasible prices, then check which prices satisfy the problem's conditions.

When we multiply each of the four prices by 100, the problem becomes finding four integers w, x, y, z for which

$$wxyz \;=\; 711000000 \qquad \text{and} \qquad w+x+y+z \;=\; 711\,.$$

Our task is to factor $711000000 = 2^6 \cdot 3^2 \cdot 5^6 \cdot 79$ into four numbers that sum to 711. Because the fourth root of the product is approximately 163, and $4 \cdot 163 = 652$, we seek factors somewhere around 163 while being aware that there is considerable leeway.

Since 79 is the largest prime factor, it makes sense to begin with that: we try letting w be a small multiple of 79. To our surprise there turns out to be no solution with $w = 2 \cdot 79 = 158$ or $w = 3 \cdot 79 = 237$. We next try $w = 4 \cdot 79 = 316$, and look for values of x, y, z that satisfy

$$xyz \;=\; 2^4 \cdot 3^2 \cdot 5^6 \qquad \text{and} \qquad x+y+z \;=\; 395\,.$$

For the sum to be divisible by 5 (and since setting $x = 5^6$ would make the sum too large), each of the three prices must be a multiple of 5. We therefore set $x = 5x'$, $y = 5y'$, and $z = 5z'$, and thereby reduce the problem to finding x', y', z' that satisfy

$$x'y'z' \;=\; 2^4 \cdot 3^2 \cdot 5^3 \qquad \text{and} \qquad x'+y'+z' \;=\; 79\,.$$

We can simplify matters further by noting that for a sum of 79, not all three unknowns can be multiples of 5; consequently, at least one of them, say x', has to be a multiple of 25. Setting $x' = 25t$, we now must solve

$$ty'z' \;=\; 2^4 \cdot 3^2 \cdot 5 \qquad \text{and} \qquad 25t+y'+z' \;=\; 79\,.$$

It is easily seen that $t \neq 2$ (since $y'z' = 2^3 \cdot 3^2 \cdot 5 = 360$ and $y'+z' = 29$ have no common solution), and also $t \neq 3$ (since $y'z' = 2^4 \cdot 3 \cdot 5 = 240$ and $y'+z' = 4$ have no common solution). We are left with $t = 1$, which gives $y'z' = 720$, and $y'+z' = 54$. This is satisfied by $y' = 24$ and $z' = 30$. Therefore, one solution is $x = 5 \cdot 1 \cdot 25 = 125$, $y = 5 \cdot 24 = 120$, and $z = 5 \cdot 30 = 150$, as claimed.

To convince oneself that the solution is unique, it seems unavoidable to investigate all multiples of 79, the approach that is outlined in the cited text. The problem is small enough that a computer can quickly check all sets of four numbers that sum to 7.11 to see if their product is 7.11. Two near misses occur: the prices \$1.25, \$1.25, \$1.44, and \$3.16 sum to \$7.10, while their product (as numbers) is 7.11; the prices \$.75, \$2.00, \$2.00, and \$2.37 sum to \$7.12, also with a product of 7.11.

M204. *Proposed by Geneviève Lalonde, Massey, ON.*

Suppose that there is a line of 2005 buttons numbered 1 through 2005. Above each button is a counter initially set to 0. Each time a button is pushed, the corresponding counter advances by 1. A set of 2005 people now proceed down the line of buttons. The first person pushes every button, the second person pushes every second button starting at button #2, the third person pushes every third button starting at button #3, and so on, so that the 2005^{th} person pushes only button #2005. When everyone has gone, which buttons' counters will read 4?

Please provide a description of the set of buttons, rather than the actual list.

Solution by the proposer.

The first button whose counter will read 4 is button #6. It is pushed by persons numbered 1, 2, 3, and 6. The numbers 1, 2, 3, and 6 are the positive divisors of 6, including 1 and the number itself. A button's counter will obviously read 4 if the number n on the button has exactly four positive divisors, including 1 and the number itself.

Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of n, where the p_is are distinct primes and the a_is are positive integers. It is well known that the number of distinct divisors of n is $(a_1+1)(a_2+1)\cdots(a_k+1)$, which includes 1 and the number itself. [The reason for this is as follows: each divisor of n has the form $p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}$, with $0 \le c_i \le a_i$ for $1 \le i \le k$. Since there are $a_i + 1$ possible values for each c_i, the formula follows. Note that if $c_i = 0$ for all $i = 1, 2, \ldots, k$, the corresponding divisor would be 1.] Hence, n has 4 divisors if and only if $(a_1+1)(a_2+1)\cdots(a_k+1) = 4$, which is true if and only if either $k = 1$ and $a_1 = 3$ or $k = 2$ and $a_1 = a_2 = 1$.

Thus, the counter of button #n will read 4 if and only if $n = p^3$, where p is a prime, or $n = pq$, where p and q are distinct primes such that $n \le 2005$.

M205. *Proposed by John Katic, Ottawa, ON.*

Show that for every triangle ABC,

$$1 \;\le\; \cos A + \cos B + \cos C \;\le\; \frac{3}{2}\,.$$

Solution by Alper Cay, Uzman Private School, Kayseri, Turkey, with details added by the editor.

Let a, b, c be the lengths of the sides opposite the angles A, B, C, respectively, and let $s = \frac{1}{2}(a+b+c)$ be the semiperimeter of $\triangle ABC$. It is known that

$$\cos A + \cos B + \cos C \;=\; 1 + 4\,\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\,.$$

(See, for example, problem 2760 [2003 : 342–343].) Furthermore, we have $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$, with similar expressions for $\sin\left(\frac{B}{2}\right)$ and $\sin\left(\frac{C}{2}\right)$. Therefore,

$$\cos A + \cos B + \cos C \;=\; 1 + 4\,\frac{(s-a)(s-b)(s-c)}{abc}\,.$$

Thus, the inequalities that we want to prove are equivalent to

$$0 \leq 4\,\frac{(s-a)(s-b)(s-c)}{abc} \leq \frac{1}{2};$$

that is,

$$0 \leq (-a+b+c)(a-b+c)(a+b-c) \leq abc. \qquad (1)$$

By the Triangle Inequality, each of the factors $-a+b+c$, $a-b+c$, and $a+b-c$ is positive. Therefore, the left inequality in (1) is true. To obtain the right inequality, we apply the AM–GM Inequality as follows:

$$\begin{aligned} \sqrt{(-a+b+c)(a-b+c)} &\leq \frac{(-a+b+c)+(a-b+c)}{2} = c, \\ \sqrt{(a-b+c)(a+b-c)} &\leq \frac{(a-b+c)+(a+b-c)}{2} = a, \\ \sqrt{(a+b-c)(-a+b+c)} &\leq \frac{(a+b-c)+(-a+b+c)}{2} = b. \end{aligned}$$

Multiplying these three inequalities gives the desired result.

Also solved by JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; and MIHÁLY BENCZE, Brasov, Romania.

M206. *Proposed by Bill Arden, Rideau High School, Ottawa, ON.*

Let n be a composite number such that $a^{n-1}-1$ is divisible by n for every number a that does not have a factor in common with n. Prove that at least 3 distinct primes divide n.

Editorial Comments: A full understanding of the solution to this innocent-looking problem is actually beyond the scope of this Column. (The editors goofed in letting this problem be printed in Mayhem!) We will instead give a brief history of the origin of this problem and the known results.

As is perhaps known to some high school students, Fermat's Little Theorem states that, if n is a prime, and a is a positive integer such that a and n are relatively prime, then $a^{n-1} \equiv 1 \pmod{n}$. This raises the natural question whether the congruence can hold if n is composite. The answer turns out to be yes, and it leads to the concept of *pseudoprimes*. Specifically, given an integer $a \geq 2$, a composite number n is called a pseudoprime to the base a if a and n are relatively prime and if $a^{n-1} \equiv 1 \pmod{n}$. For example, $n = 341 = 11 \times 31$ is a pseudoprime to the base 2, since it can be verified that $2^{340} \equiv 1 \pmod{341}$. Similarly, we can show that 91 is a pseudoprime to the base 3.

One could now ask if there is a composite n which is a pseudoprime to the base a for **all** positive integers a that are relatively prime to n. The answer, surprisingly, is still yes. Such an integer is called a *Carmichael number*, named after Robert Daniel Carmichael (1879–1967). The smallest such number is $561 = 3 \times 11 \times 17$. Another is $2821 = 7 \times 13 \times 31$.

Many interesting properties of Carmichael numbers are known. For example, a Carmichael number is square-free (a product of distinct primes) and has at least three distinct odd prime divisors (this is exactly the problem M206). In fact, a complete characterization of a Carmichael number is known: namely, n is a Carmichael number if and only if $n = p_1 p_2 \cdots p_k$ where the p_is are distinct primes such that $(p_i - 1) \mid (n - 1)$ for all $i = 1, 2, \ldots, k$.

In 1912 Carmichael conjectured that there exist infinitely many such numbers. This was a very difficult conjecture, and it took 80 years before it was finally proved in the affirmative by the following three mathematicians: Alford, Granville, and Pomerance.

Readers interested in this topic are encouraged to consult the references listed below. Most of the information given above can be found in [1].

A complete solution to the current problem was submitted by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina. His proof, however, is quite technical and used, among other things, known results from number theory about primitive roots.

References

[1] Kenneth H. Rosen, *Elementary Number Theory and its Applications*, 5^{th} ed., Addison Wesley and Longman, 2005.

[2] W. Sierpinski, *Elementary Theory of Numbers*, 2^{nd} ed., North-Holland, Arsterdam, 1987.