# MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is Jeff Hooper (Acadia University). The other staff members are John Grant McLoughlin (University of New Brunswick), Ian VanderBurgh (University of Waterloo), Larry Rice (University of Waterloo), Eric Robert (Leo Hayes High School, Fredericton), Monika Khbeis (Father Michael Goetz Secondary School, Mississauga), Mark Bredin (St. John's-Ravenscourt School, Winnipeg), and Ron Lancaster (University of Toronto).

## Mayhem Problems

*Please send your solutions to the problems in this edition by* **1 January 2007**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

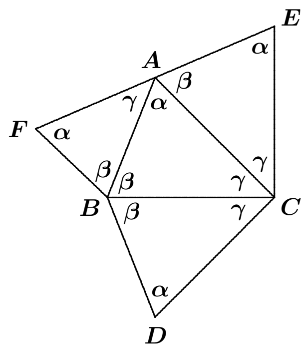
*Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.*

*The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.*

**M251**. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let $\alpha$, $\beta$, $\gamma$ be the angle measures at angles $A$, $B$, $C$, respectively, in $\triangle ABC$. On the sides of $\triangle ABC$, externally, are triangles $DBC$, $EAC$, and $FBA$ as in the diagram.

Prove that $AD = EF$ if and only if $\alpha = \pi/2$.




**M252**. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $x$, $y$, $z$ be positive real numbers. Prove that

$$\left(\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}}\right)^2+\left(\frac{y}{z}+\frac{x}{\sqrt[3]{xyz}}\right)^2+\left(\frac{z}{x}+\frac{y}{\sqrt[3]{xyz}}\right)^2 \;\geq\; 12\,.$$

**M253**. *Proposed by Fabio Zucca, Politecnico di Milano, Milano, Italy.*

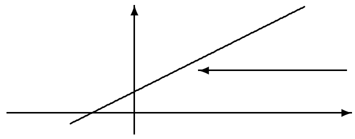
Consider the set of lattice points $\{(x, y)\}$ where $x$ and $y$ are integers such that $0 \leq x \leq 7$ and $0 \leq y \leq 7$. Two points are selected at random from this set. All points have the same probability of being selected and the points need not be distinct. Find the probability that the area of the triangle (possibly degenerate) formed by these two points and the point $(0, 0)$ is an integer (possibly 0).

**M254**. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Evaluate the summation $S_{2006} = \sum\limits_{k=1}^{2006} (-1)^k \frac{k^2-3}{(k+1)!}$. [Recall that $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$; for example, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.]

**M255**. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The line with slope $\lambda > 0$ acts like a mirror to a ray of light coming along a line parallel to the $x$–axis. Determine the slope of the reflected ray.



**M256**. *Proposed by the Mayhem Staff.*

Find a quadratic polynomial $f(x)$ such that, if $n$ is a positive integer consisting of the digit 5 repeated $k$ times, then $f(n)$ consists of the digit 5 repeated $2k$ times. (For example, $f(555) = 555555$.)

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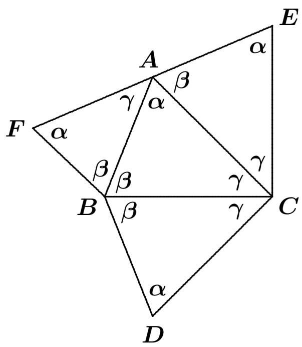
**M251**. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Soit $\alpha$, $\beta$ et $\gamma$ les mesures respectives des angles $A$, $B$ et $C$ dans le triangle $ABC$. Sur les côtés du triangle $ABC$, on construit extérieurement les triangles $DBC$, $EAC$ et $FBA$, comme indiqué dans la figure.

Montrer que $AD = EF$ si et seulement si $\alpha = \pi/2$.




**M252**. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $x$, $y$ et $z$ trois nombres réels positifs. Montrer que

$$\left(\frac{x}{y} + \frac{z}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{y}{z} + \frac{x}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{z}{x} + \frac{y}{\sqrt[3]{xyz}}\right)^2 \geq 12\,.$$

**M253**. *Proposé par Fabio Zucca, Politecnico di Milano, Milano, L'italie.*

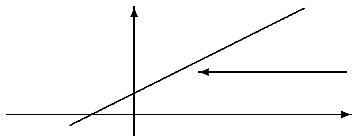
On considère l'ensemble des points $\{(x, y)\}$ d'un réseau où $x$ et $y$ sont des entiers tels que $0 \leq x \leq 7$ et $0 \leq y \leq 7$. On choisit deux points de cet ensemble au hasard. Tous les points ont la même probabilité d'être choisis et les points peuvent ne pas être distincts. Trouver la probabilité pour que l'aire du triangle (peut-être dégénéré) formé par ces points et le point $(0, 0)$ soit un entier (peut-être 0).

**M254**. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Evaluer la somme $S_{2006} = \sum_{k=1}^{2006} (-1)^k \frac{k^2 - 3}{(k+1)!}$. [On rappelle que $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$ ; par exemple, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.]

**M255**. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Une droite de pente $\lambda > 0$ agit comme un miroir sur un rayon lumineux suivant une droite parallèle à l'axes des $x$. Déterminer la pente du rayon réfléchi.



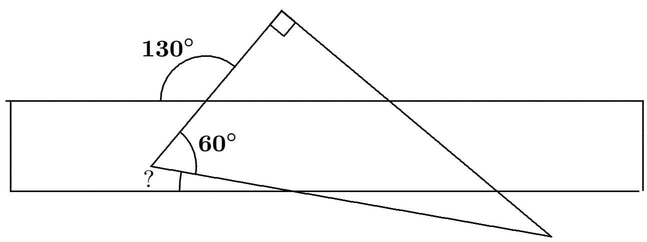
**M256**. *Proposé par l'Équipe de Mayhem.*

Trouver un polynôme quadratique $f(x)$ tel que, si $n$ est un entier positif formé du chiffre 5 répété $k$ fois, alors $f(n)$ est formé du chiffre 5 répété $2k$ fois. (Par exemple, $f(555) = 555555$.)

# Mayhem Solutions

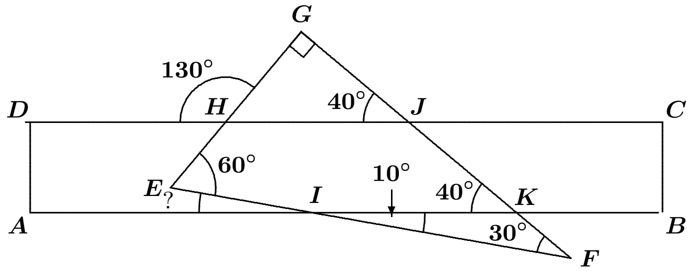
**M201**. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

A student drops his $30^\circ$–$60^\circ$–$90^\circ$ triangle on his ruler so that a $130^\circ$ angle appears as in the diagram below. What is the measure of the other marked angle?



*Solution by Glenier L. Bello-Burguet, student, 3ro de ESO Instituto Hermanos D'Elhuyar, Logroño, Spain.*

We label the diagram as shown. The angle we seek is $\angle AIE = 10^\circ$.




We are given $\angle GHD = 130^\circ$ and also the angles of triangle $EFG$, namely $\angle GEF = 60^\circ$, $\angle FGE = 90^\circ$, $\angle EFG = 30^\circ$. Thus, $\angle HJG = 40^\circ$ (since $\angle HJG + \angle HGJ = \angle GHD$), and then the corresponding $\angle JKI$ also equals $40^\circ$. Hence, $\angle KIF = 10^\circ$ (because $\angle KIF + \angle EFG = \angle JKI$) and, therefore, also $\angle AIE = 10^\circ$.

*Also solved by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.*

**M202**. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

(a) By the end of the season, Samuel had made more than 50% of his foul shots, even though at the start of the season his average was below 50%. Show that there was a time during the season when his average was exactly 50%.

(b) For what other percentages $p$ can you be certain that the average was exactly $p$ at some time when you know only that the average was below $p$ and then above $p$ at a later time?

*Solution by the Mayhem Staff.*

(a) For the purpose of contradiction, assume that there was no time during the season when Samuel's average was exactly 50%. Then there must have been a time when, by sinking a foul shot, he increased his average from a fraction $a/b < 1/2$ to $(a+1)/(b+1) > 1/2$, where $a$ and $b$ are integers with $a \geq 0$ and $b > 0$. Then $2a < b$ and $2a + 2 > b + 1$. Together, these inequalities imply that $2a + 1 < b + 1 < 2a + 2$, which is impossible, since the integer $b + 1$ cannot lie strictly between two consecutive integers. This contradiction establishes the existence of a point in the season when Samuel had an average of 50%.

(b) By generalizing the argument in part (a) (using a fraction $c/d$ in place of $1/2$), we can show that the percentages we seek are those of the form $c/(c+1)$, where $c$ is a positive integer.

[*Ed:* This problem was explored in the Problem of the Month feature in the April 2006 issue of ***Crux Mathematicorum*** [2], and there is even a Proof Without Words treatment which has recently appeared [1].]

**References**


[1] Robert J. MacG. Dawson, Putnam Proof Without Words, *Mathematics Magazine*, 79 (2006), p. 149.

[2] Ian VanderBurgh, Problem of the Month, *Crux Mathematicorum with Mathematical Mayhem*, 32 (2006), pp. 147–148.


**M203**. *Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

Chuck goes into the local 7-11 store and buys four items. The bill totals $\$7.11$. He notices that the product of the four prices is exactly $7.11$. What are the prices of the four items?

[*Ed:* The proposer has indicated that the problem does not originate with him, nor does he know its origin.]

*Solution by the University of Regina Problem Group.*

The items cost $\$1.20$, $\$1.25$, $\$1.50$, and $\$3.16$. We found the problem in *How To Solve It: Modern Heuristics*, by Zbigniew Michalewicz and David B. Fogel (Springer, 1998), Puzzle III, pages 49–53. The problem seems not to have originated with these authors, because they refer to it as the "so-called '7–11' problem", but they do not provide a reference. Our approach was essentially the same as theirs: convert the prices into cents (so that we can use integer arithmetic), systematically list all possible combinations of feasible prices, then check which prices satisfy the problem's conditions.

When we multiply each of the four prices by $100$, the problem becomes finding four integers $w$, $x$, $y$, $z$ for which

$$wxyz \;=\; 711000000 \qquad \text{and} \qquad w+x+y+z \;=\; 711\,.$$

Our task is to factor $711000000 = 2^6 \cdot 3^2 \cdot 5^6 \cdot 79$ into four numbers that sum to $711$. Because the fourth root of the product is approximately $163$, and $4 \cdot 163 = 652$, we seek factors somewhere around $163$ while being aware that there is considerable leeway.

Since $79$ is the largest prime factor, it makes sense to begin with that: we try letting $w$ be a small multiple of $79$. To our surprise there turns out to be no solution with $w = 2 \cdot 79 = 158$ or $w = 3 \cdot 79 = 237$. We next try $w = 4 \cdot 79 = 316$, and look for values of $x$, $y$, $z$ that satisfy

$$xyz \;=\; 2^4 \cdot 3^2 \cdot 5^6 \qquad \text{and} \qquad x+y+z \;=\; 395\,.$$

For the sum to be divisible by $5$ (and since setting $x = 5^6$ would make the sum too large), each of the three prices must be a multiple of $5$. We therefore set $x = 5x'$, $y = 5y'$, and $z = 5z'$, and thereby reduce the problem to finding $x'$, $y'$, $z'$ that satisfy

$$x'y'z' \;=\; 2^4 \cdot 3^2 \cdot 5^3 \qquad \text{and} \qquad x' + y' + z' \;=\; 79\,.$$

We can simplify matters further by noting that for a sum of 79, not all three unknowns can be multiples of 5; consequently, at least one of them, say $x'$, has to be a multiple of 25. Setting $x' = 25t$, we now must solve

$$ty'z' \;=\; 2^4 \cdot 3^2 \cdot 5 \qquad \text{and} \qquad 25t + y' + z' \;=\; 79\,.$$

It is easily seen that $t \neq 2$ (since $y'z' = 2^3 \cdot 3^2 \cdot 5 = 360$ and $y' + z' = 29$ have no common solution), and also $t \neq 3$ (since $y'z' = 2^4 \cdot 3 \cdot 5 = 240$ and $y' + z' = 4$ have no common solution). We are left with $t = 1$, which gives $y'z' = 720$, and $y' + z' = 54$. This is satisfied by $y' = 24$ and $z' = 30$. Therefore, one solution is $x = 5 \cdot 1 \cdot 25 = 125$, $y = 5 \cdot 24 = 120$, and $z = 5 \cdot 30 = 150$, as claimed.

To convince oneself that the solution is unique, it seems unavoidable to investigate all multiples of 79, the approach that is outlined in the cited text. The problem is small enough that a computer can quickly check all sets of four numbers that sum to 7.11 to see if their product is 7.11. Two near misses occur: the prices \$1.25, \$1.25, \$1.44, and \$3.16 sum to \$7.10, while their product (as numbers) is 7.11; the prices \$.75, \$2.00, \$2.00, and \$2.37 sum to \$7.12, also with a product of 7.11.

## M204. *Proposed by Geneviève Lalonde, Massey, ON.*

Suppose that there is a line of 2005 buttons numbered 1 through 2005. Above each button is a counter initially set to 0. Each time a button is pushed, the corresponding counter advances by 1. A set of 2005 people now proceed down the line of buttons. The first person pushes every button, the second person pushes every second button starting at button #2, the third person pushes every third button starting at button #3, and so on, so that the $2005^{\text{th}}$ person pushes only button #2005. When everyone has gone, which buttons' counters will read 4?

Please provide a description of the set of buttons, rather than the actual list.

*Solution by the proposer.*

The first button whose counter will read 4 is button #6. It is pushed by persons numbered 1, 2, 3, and 6. The numbers 1, 2, 3, and 6 are the positive divisors of 6, including 1 and the number itself. A button's counter will obviously read 4 if the number $n$ on the button has exactly four positive divisors, including 1 and the number itself.

Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of $n$, where the $p_i$s are distinct primes and the $a_i$s are positive integers. It is well known that the number of distinct divisors of $n$ is $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$, which includes 1 and the number itself. [The reason for this is as follows: each divisor of $n$ has the form $p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}$, with $0 \le c_i \le a_i$ for $1 \le i \le k$. Since there are $a_i + 1$ possible values for each $c_i$, the formula follows. Note that if $c_i = 0$ for all $i = 1, 2, \ldots, k$, the corresponding divisor would be 1.] Hence, $n$ has 4 divisors if and only if $(a_1+1)(a_2+1)\cdots(a_k+1) = 4$, which is true if and only if either $k = 1$ and $a_1 = 3$ or $k = 2$ and $a_1 = a_2 = 1$.

Thus, the counter of button $\#n$ will read 4 if and only if $n = p^3$, where $p$ is a prime, or $n = pq$, where $p$ and $q$ are distinct primes such that $n \le 2005$.

**M205**. *Proposed by John Katic, Ottawa, ON.*

Show that for every triangle $ABC$,

$$1 \le \cos A + \cos B + \cos C \le \frac{3}{2}\,.$$

*Solution by Alper Cay, Uzman Private School, Kayseri, Turkey, with details added by the editor.*

Let $a$, $b$, $c$ be the lengths of the sides opposite the angles $A$, $B$, $C$, respectively, and let $s = \frac{1}{2}(a+b+c)$ be the semiperimeter of $\triangle ABC$. It is known that

$$\cos A + \cos B + \cos C = 1 + 4 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right)\,.$$

(See, for example, problem 2760 [2003 : 342–343].) Furthermore, we have $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$, with similar expressions for $\sin\left(\frac{B}{2}\right)$ and $\sin\left(\frac{C}{2}\right)$. Therefore,

$$\cos A + \cos B + \cos C = 1 + 4\,\frac{(s-a)(s-b)(s-c)}{abc}\,.$$

Thus, the inequalities that we want to prove are equivalent to

$$0 \le 4\,\frac{(s-a)(s-b)(s-c)}{abc} \le \frac{1}{2}\,;$$

that is,

$$0 \le (-a+b+c)(a-b+c)(a+b-c) \le abc\,. \qquad (1)$$

By the Triangle Inequality, each of the factors $-a+b+c$, $a-b+c$, and $a+b-c$ is positive. Therefore, the left inequality in (1) is true. To obtain the right inequality, we apply the AM–GM Inequality as follows:

$$\begin{aligned}
\sqrt{(-a+b+c)(a-b+c)} &\le \frac{(-a+b+c)+(a-b+c)}{2} = c\,,\\
\sqrt{(a-b+c)(a+b-c)} &\le \frac{(a-b+c)+(a+b-c)}{2} = a\,,\\
\sqrt{(a+b-c)(-a+b+c)} &\le \frac{(a+b-c)+(-a+b+c)}{2} = b\,.
\end{aligned}$$

Multiplying these three inequalities gives the desired result.

*Also solved by JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; and MIHÁLY BENCZE, Brasov, Romania.*

**M206**. *Proposed by Bill Arden, Rideau High School, Ottawa, ON.*

Let $n$ be a composite number such that $a^{n-1} - 1$ is divisible by $n$ for every number $a$ that does not have a factor in common with $n$. Prove that at least $3$ distinct primes divide $n$.

*Editorial Comments:* A full understanding of the solution to this innocent-looking problem is actually beyond the scope of this Column. (The editors goofed in letting this problem be printed in Mayhem!) We will instead give a brief history of the origin of this problem and the known results.

As is perhaps known to some high school students, Fermat's Little Theorem states that, if $n$ is a prime, and $a$ is a positive integer such that $a$ and $n$ are relatively prime, then $a^{n-1} \equiv 1 \pmod{n}$. This raises the natural question whether the congruence can hold if $n$ is composite. The answer turns out to be yes, and it leads to the concept of *pseudoprimes*. Specifically, given an integer $a \geq 2$, a composite number $n$ is called a pseudoprime to the base $a$ if $a$ and $n$ are relatively prime and if $a^{n-1} \equiv 1 \pmod{n}$. For example, $n = 341 = 11 \times 31$ is a pseudoprime to the base $2$, since it can be verified that $2^{340} \equiv 1 \pmod{341}$. Similarly, we can show that $91$ is a pseudoprime to the base $3$.

One could now ask if there is a composite $n$ which is a pseudoprime to the base $a$ for **all** positive integers $a$ that are relatively prime to $n$. The answer, surprisingly, is still yes. Such an integer is called a *Carmichael number*, named after Robert Daniel Carmichael (1879–1967). The smallest such number is $561 = 3 \times 11 \times 17$. Another is $2821 = 7 \times 13 \times 31$.

Many interesting properties of Carmichael numbers are known. For example, a Carmichael number is square-free (a product of distinct primes) and has at least three distinct odd prime divisors (this is exactly the problem M206). In fact, a complete characterization of a Carmichael number is known: namely, $n$ is a Carmichael number if and only if $n = p_1 p_2 \cdots p_k$ where the $p_i$s are distinct primes such that $(p_i - 1) \mid (n - 1)$ for all $i = 1, 2, \ldots, k$.

In 1912 Carmichael conjectured that there exist infinitely many such numbers. This was a very difficult conjecture, and it took 80 years before it was finally proved in the affirmative by the following three mathematicians: Alford, Granville, and Pomerance.

Readers interested in this topic are encouraged to consult the references listed below. Most of the information given above can be found in [1].

A complete solution to the current problem was submitted by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina. His proof, however, is quite technical and used, among other things, known results from number theory about primitive roots.

**References**

[1] Kenneth H. Rosen, *Elementary Number Theory and its Applications*, 5th ed., Addison Wesley and Longman, 2005.

[2] W. Sierpinski, *Elementary Theory of Numbers*, 2nd ed., North-Holland, Arsterdam, 1987.

# Problem of the Month

Ian VanderBurgh

Next in the sequence of Problems of the Month comes a problem (actually, two problems) about sequences.

**Problem 1** (2006 Pascal Contest)

John writes a number with 2187 digits on the blackboard, each digit being a 1 or a 2. Judith creates a new number from John's number by reading his number from left to right and wherever she sees a 1, writing 112, and wherever she sees a 2, writing 111. (For example, if John's number begins 2112, then Judith's number would begin 111112112111.) After Judith finishes writing her number, she notices that the left-most 2187 digits in her number and in John's number are the same. How many times do five 1s occur consecutively in John's number?

This is quite something to try to wrap your head around! Let's start off with a slightly different problem, which is a bit easier to sort out:

**Problem 2**. John writes a number with 3000 digits on the blackboard, each digit being a 1 or a 2. Judith creates a new number from John's number by reading his number from left to right and wherever she sees a 1, writing 122, and wherever she sees a 2, writing 111. (For example, if John's number begins 2112, then Judith's number would begin 111122122111.) After Judith finishes writing her number, she notices that the left-most 2500 digits in her number and in John's number are the same. What are the digits in positions 2006 to 2012? (Problem 2 appeared on one of the earlier drafts of the 2006 Pascal Contest, but was removed in favour of Problem 1.)

It is tough to know where to start this problem; as the song says, though, let's start at the very beginning (it's a very good place to start).

*Solution to Problem 2:* If John's number begins with a 1, then Judith's will begin 122; if John's number begins with a 2, Judith's will begin 111. Since Judith's number begins with a 1 regardless of what John's number begins with, and since their numbers are the same for the first 2500 digits, John's number must begin with a 1. Thus, Judith's begins 122, and therefore John's begins 122. (We call these three digits Stage 1.)

We can then use this beginning as the "seed" with which to create the number. Since John's number begins 122, then Judith's begins 122111111 (since each 1 becomes 122 and each 2 becomes 111). Thus, John's number begins 122111111 (Stage 2).

Continuing in a similar way, each digit at each Stage generates three digits at the next Stage, which means that each Stage is three times as long as the previous one. Thus, successive stages consist of 3, 9, 27, 81, 243, 729, and 2187 digits. In general, Stage $n$ consists of $3^n$ digits.

Consider Stage 7, which has length $3^7 = 2187$. This includes the digits in positions 2006 to 2012, the positions that interest us. How can we figure out what these digits are? One strategy is to trace backwards through the Stages the positions of the digits which generate these digits, until we come to one of the early Stages, where we can easily determine the digits. We then use these known digits to move forward again through the Stages to determine the desired digits at Stage 7.

In general, the first $k$ digits from one Stage generate the first $3k$ digits at the next stage. This tells us that digit $k$ from one Stage generates digits $3k-2$, $3k-1$, and $3k$ at the next stage. We would like digits 2006 to 2012 at Stage 7. Digit 669 from Stage 6 generates digits 2005 to 2007 at Stage 7, digit 670 generates digits 2008 to 2010 at Stage 7, and digit 671 generates digits 2011 to 2013 at Stage 7. If we can determine digits 669 to 671 at Stage 6, then we can get digits 2006 to 2012 at Stage 7. (These digits at Stage 6 actually will give us digits 2005 to 2013 at Stage 7, which is a bit more than we need.)

Perhaps a chart might be in order:

| Stage | Digit(s) | | Stage | Digit(s) | | Digit(s) |
|---|---|---|---|---|---|---|
| 7 | 2006–2012 | from | 6 | 669–671 | giving | 2005–2013 |
| 6 | 669–671 | from | 5 | 223–224 | giving | 667–672 |
| 5 | 223–224 | from | 4 | 75 | giving | 223–225 |
| 4 | 75 | from | 3 | 25 | giving | 73–75 |
| 3 | 25 | from | 2 | 9 | giving | 25–27 |

Aha! We know that digit number 9 at Stage 2 is a 1. Hence, remembering that a 1 at one stage gives 122 at the next stage and a 2 at one stage gives 111 at the next stage, we can then trace these digits forward through the Stages. Probably making another table will help:

| Stage | Digit(s) | String | | Stage | Digits | String |
|---|---|---|---|---|---|---|
| 2 | 9 | 1 | gives | 3 | 25–27 | 122 |
| 3 | 25 | 1 | gives | 4 | 73–75 | 122 |
| 4 | 75 | 2 | gives | 5 | 223–225 | 111 |
| 5 | 223–224 | 11 | gives | 6 | 667–672 | 122122 |
| 6 | 669–671 | 212 | gives | 7 | 2005–2013 | 111122111 |

Thus, the digits in positions 2006 to 2012 are 1112211.

Making a table seemed to be pretty helpful here. Often, it is a really good way to consolidate or organize a bunch of information. Now Problem 1 certainly looks similar (be careful, though—the digit replacement scheme is slightly different).

*Solution to Problem 1:* Conveniently, here we are looking at 2187 digits, which exactly matches the length of Stage 7 in our analysis above. (This seems unlikely to be a coincidence!) Here, the digit replacement scheme is that 1 becomes 112 and 2 becomes 111. Again, John's number must start with a 1, and thus must start 112 (Stage 1).

How can 11111 (that is, five consecutive 1s) appear at Stage 7? They must come from 21 at Stage 6. They cannot come from 22 because there cannot be two consecutive 2s at any stage. (Think about why this is true.) In fact, there cannot be more than five consecutive 1s at any stage. (Think about this as well.)

Thus, counting the number of occurrences of 11111 at Stage 7 is the same as counting the number of occurrences of 21 at Stage 6. But each 2 is always followed by a 1; whence, this is the same as the number of 2s at Stage 6. (Wait! If the 2 occurs at the very end of Stage 6, then it is not followed by a 1. It turns out that Stage 6 ends in a 1—a third thing to think about!)

How can we figure out the number of 2s at Stage 6? The only way to get a 2 at Stage 6 is from a 1 at Stage 5, which gives 112. (And each 1 at Stage 5 gives exactly one 2 at Stage 6).

At this point, trying to write all of this out in words is becoming painful. Thus, some notation would be helpful. Let's use $a_n$ to represent the number of 1s at Stage $n$ and $b_n$ to represent the number of 2s at Stage $n$. Notice that $a_1 = 2$ and $b_1 = 1$. Also, $b_{n+1} = a_n$ (each 1 at Stage $n$ generates exactly one 2 at Stage $n + 1$) and $a_{n+1} = 2a_n + 3b_n$ (each 1 at Stage $n$ generates two 1s and each 2 generates three 1s). Again, a table:

| $n$ | $a_n$ | $b_n$ |
|---|---|---|
| 1 | 2 | 1 |
| 2 | 7 | 2 |
| 3 | 20 | 7 |
| 4 | 61 | 20 |
| 5 | 182 | 61 |

Therefore, the number of 1s at Stage 5 is 182, which implies that there are 182 occurrences of 11111 at Stage 7 (that is, in the first 2187 digits of the full number).

Yes, this problem was pretty difficult to be on a Grade 9 Contest(!). But it required very little specialized knowledge, and lots could be discovered through some fiddling, which was why we felt comfortable including it. And there is a lot more that could be asked, and lots of interesting investigation to be done (including lots that could be investigated with a computer). Happy hunting! It is also interesting to look at what happens by modifying the digit replacement schemes, even allowing the 1 and 2 to be replaced by strings of different lengths.

# Pólya's Paragon

## Playing Games with Mathematics (Part I)

John Grant McLoughlin

Recreational mathematics provides a wonderful vehicle for developing mathematical thinking. Game-playing arouses curiosity about the principles underlying the structures and strategies associated with the games. This is particularly evident when the players are students at a mathematical event such as the University of New Brunswick Math Camp. This column and its successor are based on a recent contribution to that event, in May 2006. Four of the games and challenges that were posed at the camp are shared here, followed by an additional problem involving polynomials.

The purpose of the challenges is threefold: to broaden one's knowledge of recreational mathematics; to engage in playful mathematical activity; and to develop mathematical arguments and proofs in response to the challenges. This latter point will be the focus of Part II, in which these challenges are to be discussed. It should be noted that this two-part model was used at the camp also. The challenges were shared one afternoon in an introductory session, and the discussion of the mathematical principles, including insights gained from participating students, took place the next evening.

**1. Sim**

Six dots are drawn on a piece of paper to form the vertices of a hexagon. Two players are each assigned a colour. The players take turns joining any two of the dots with a line segment, using their assigned colours. The loser is the player who completes a triangle with three of the original six dots as its vertices and with all three edges the same colour.

*Challenge:* Prove that there must always be a loser (and a winner).

**2. 31**

This mental math game involves a running total which starts at zero. Each player has the choice to add 1, 2, 3, 4, 5, or 6 to the total. Players alternate turns. The winner is the player who is able to bring the total to 31.

For example, player B is the winner in the following game:

| Player | A | B | A | B | A | B | A | B |
|---|---|---|---|---|---|---|---|---|
| Chosen number | 6 | 4 | 2 | 4 | 4 | 1 | 5 | 5 |
| Total | 6 | 10 | 12 | 16 | 20 | 21 | 26 | 31 |

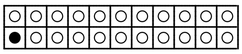
*Challenge:* Determine a winning strategy. You may choose to play first or second.

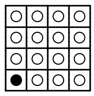
**3. Chomp**

Counters are placed in a rectangular grid such that one counter appears in each small rectangle. The counter in the bottom left-hand corner is a different colour than the others. Players take turns selecting one counter. If the counter selected occupies the bottom left-hand corner of a rectangle on the grid, all the counters in that rectangle are removed. The object is to force your opponent to select the differently coloured counter (the one in the bottom left-hand corner).

Try playing this game with rectangular boards of different sizes.

*Challenge:* Suppose that you play two games of Chomp in which the boards are $2 \times n$ and $k \times k$, examples of which are shown. Determine a winning strategy in each case. You may choose to play first or second.





**4. Fifteen Finesse**

The numbers $1$, $2$, $3$, $4$, $5$, $6$, $7$, $8$, and $9$ are available for use in this game. Each number can be used only once. Two players alternate turns selecting one of the available numbers. To win the game, a player must obtain exactly three numbers that sum to $15$. (Neither a pair of numbers, such as $7$ and $8$, nor a set of four numbers, such as $1$, $3$, $5$, and $6$, constitutes a winning combination.) The game ends in a draw if no player is able to acquire three numbers that sum to $15$.

*Challenge:* Explain the underlying structure of the game, and suggest strategies that may help to win a game.

**5. A Polynomial in Transition**

Consider the polynomial $x^2 + 10x + 20$. Under the conditions below, is it possible to convert this polynomial to $x^2 + 20x + 10$? Justify your answer.

Conditions:

(i) On each step you may only change the constant term or the coefficient of $x$ (but not both).

(ii) The change must be an increase of $1$ or a decrease of $1$.

(iii) The change must NOT produce a polynomial that can be factored into the form $(x+m)(x+n)$ where $m$ and $n$ are integers. For example, you could not begin by reducing $10$ to $9$, since $x^2+9x+20 = (x+5)(x+4)$.

Various sources have contributed ideas/games for this presentation, including books by Brian Bolt, Ian Stewart, Martin Gardner, and others. The polynomial problem is due to Ed Barbeau. A bibliography will be included in Part II. As usual, comments on any aspect of Pólya's Paragon are welcomed.