SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

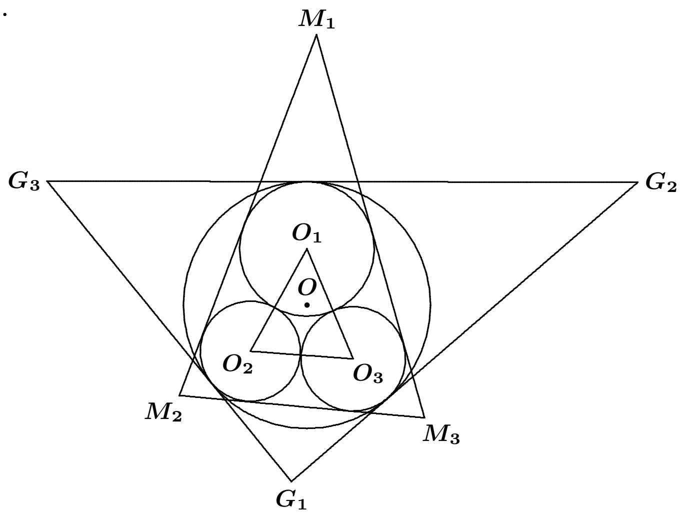
We apologize for omitting the name of Michel Bataille, Rouen, France from the list of solvers of 3028; and for omitting the name of Chip Curtis, Missouri Southern State University, Joplin, MO, USA from the list of solvers of 3039.

1150★. [1986 : 108; 1987 : 264; 1988 : 46] *Proposed by Jack Garfunkel (deceased).*

In the figure, $\triangle M_1M_2M_3$ and the three circles with centres O_1, O_2, O_3 represent the Malfatti configuration. Circle O is externally tangent to these three circles, and the sides of $\triangle G_1G_2G_3$ are each tangent to O and one of the smaller circles. Prove that

$$P(\triangle G_1G_2G_3) \;\geq\; P(\triangle M_1M_2M_3) + P(\triangle O_1O_2O_3)\,,$$

where P stands for perimeter. Equality is attained when $\triangle O_1O_2O_3$ is equilateral.



Solution by Kai-Xian Wang, Qingdao, Shandong, China, summarized by the editor.

Let r be the radius of circle O and let r_i be the radius of circle O_i for $i = 1, 2, 3$. To construct counterexamples, we set $r_2 = r_3 = 2$ and let r_1 vary: set $r_1 = r > 0$. To simplify the notation, we let $g = P(\triangle G_1G_2G_3)$, $m = P(\triangle M_1M_2M_3)$, and $o = P(\triangle O_1O_2O_3)$. Note that $o = 8 + 2r$.

Counterexample 1. When r increases to 4 (see Figure 1), g and o remain bounded while m goes to infinity. Then $g < m + o$, contrary to the conjecture. [*Ed:* This is the essence of the counterexample given in [1988 : 46].]

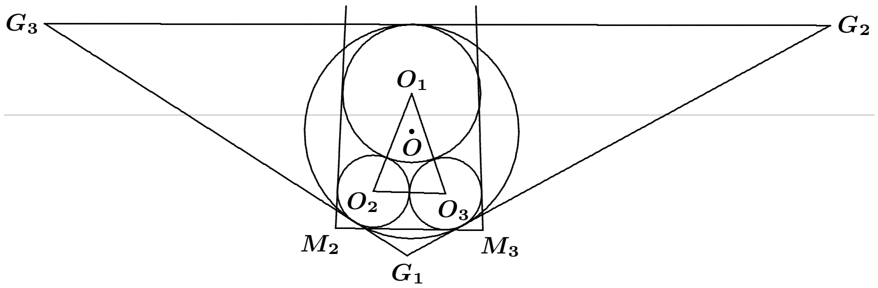


Figure 1: $r = 4 - \varepsilon$, $r_2 = r_3 = 2$

Counterexample 2. When r decreases to $\frac{1}{2}$ (see Figure 2), the radius of circle O becomes infinite as circle O_1 sinks below the horizontal common tangent to circles O_2 and O_3. Here g is bounded near 8 while m goes to infinity; thus, again $g < m + o$, contrary to the conjecture.

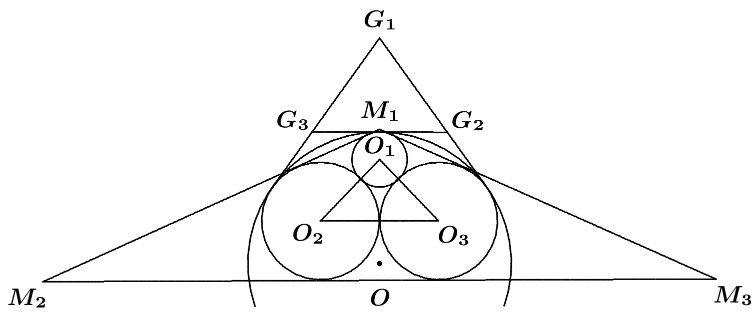


Figure 2: $r = \frac{1}{2} + \varepsilon$, $r_2 = r_3 = 2$

Wang provided explicit details for the case $r = 1$. He showed that $16.9 < g < 17.0$, $42.00 < m < 42.01$, and $o = 10$. The details are straightforward, but the computation requires about a page.

It is clear that there are values of r for which the conjectured inequality holds; for example, $g > m + o$ when the outer circle O has its centre on the line O_2O_3 (since m and o are bounded while g is infinite). Also, it is easily checked that when $r = 2$ (and all three triangles are equilateral), $g = m + o$ since $o = 12$, $m = 12 + 12\sqrt{3}$, and $g = 24 + 12\sqrt{3}$.

The statement of the conjecture contains a small error: it is clear from the figure that circle O should be internally *tangent to the three smaller circles (instead of* externally *tangent as stated). Wang first saw the conjecture among 152 open problems in Kuang Jichang,* Applied Inequalities, *3rd edition (in Chinese), Shandong Science and Technology Press, Jinan, P.R. China (2004), page 706.*

3044. [2005 : 238, 240] *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $\{a_n\}$ be the sequence defined by $a_0 = 1$, $a_1 = 2$, and, for $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$. Find the sum

$$\sum_{n=1}^{\infty} \frac{a_{2n+2}}{a_{n-1}^2 a_{n+1}^2}.$$

Essentially the same solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Note that $a_n = f_{n+2}$ for $n \geq 0$, where $\{f_n\}$ is the Fibonacci sequence. It is well known (and easily proved by induction) that for $n \geq 0$

$$\begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n .$$

It follows that

$$\begin{pmatrix} f_{2n+5} & f_{2n+4} \\ f_{2n+4} & f_{2n+3} \end{pmatrix} = \begin{pmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{pmatrix} \begin{pmatrix} f_{n+4} & f_{n+3} \\ f_{n+3} & f_{n+2} \end{pmatrix} .$$

Equating the upper-right entries, we get

$$\begin{aligned} f_{2n+4} &= f_{n+2}f_{n+3} + f_{n+1}f_{n+2} = f_{n+2}(f_{n+1} + f_{n+3}) \\ &= (f_{n+3} - f_{n+1})(f_{n+1} + f_{n+3}) = f_{n+3}^2 - f_{n+1}^2 . \end{aligned}$$

Thus

$$\frac{a_{2n+2}}{a_{n-1}^2 a_{n+1}^2} = \frac{f_{2n+4}}{f_{n+1}^2 f_{n+3}^2} = \frac{f_{n+3}^2 - f_{n+1}^2}{f_{n+1}^2 f_{n+3}^2} = \frac{1}{f_{n+1}^2} - \frac{1}{f_{n+3}^2} .$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_{2n+2}}{a_{n-1}^2 a_{n+1}^2} &= \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{f_{n+1}^2} - \frac{1}{f_{n+3}^2} \right) \\ &= \lim_{N\to\infty} \left(\frac{1}{f_2^2} + \frac{1}{f_3^2} - \frac{1}{f_{N+2}^2} - \frac{1}{f_{N+3}^2} \right) \\ &= \frac{1}{f_2^2} + \frac{1}{f_3^2} = 1 + \frac{1}{4} = \frac{5}{4} . \end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Note that the matrix representation of the Fibonacci numbers also occurs in this issue in the solution of KLAMKIN–04 on pages 314–315.

3045. [2005 : 238, 240] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c be positive real numbers such that $abc \geq 1$. Prove that

(a) $a^{\frac{a}{b}} b^{\frac{b}{c}} c^{\frac{c}{a}} \geq 1$;   (b) $a^{\frac{a}{b}} b^{\frac{b}{c}} c^{c} \geq 1$.

Solution by the proposer.

(a) First we prove the inequality in the case where $abc = 1$. The inequality may be written equivalently as $\frac{a}{b}\ln a + \frac{b}{c}\ln b + \frac{c}{a}\ln c \geq 0$. Since the function $f(x) = x\ln x$ is convex, Jensen's Inequality gives

$$\frac{1}{b}\cdot a\ln a + \frac{1}{c}\cdot b\ln b + \frac{1}{a}\cdot c\ln c \;\geq\; \left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)\cdot\ln\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{\frac{1}{b}+\frac{1}{c}+\frac{1}{a}}\,,$$

and it remains to show that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \;\geq\; \frac{1}{b}+\frac{1}{c}+\frac{1}{a}\,.$$

The last inequality is known (see problem 2886 [2003 : 468; 2004 : 518]).

Now we turn to the general case, where $abc \geq 1$. Let $x = ar$, $y = br$ and $z = cr$, where $r = \dfrac{1}{\sqrt[3]{abc}} \leq 1$. Then $xyz = 1$, and thus $x^{\frac{x}{y}}y^{\frac{y}{z}}z^{\frac{z}{x}} \geq 1$ (applying the special case we have already proved). Then

$$a^{\frac{a}{b}}b^{\frac{b}{c}}c^{\frac{c}{a}} \;\geq\; a^{\frac{a}{b}}b^{\frac{b}{c}}c^{\frac{c}{a}}r^{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}} \;=\; x^{\frac{x}{y}}y^{\frac{y}{z}}z^{\frac{z}{x}} \;\geq\; 1\,.$$

(b) We write the inequality in the form $\frac{a}{b}\ln a + \frac{b}{c}\ln b + c\ln c \geq 0$. As above, by Jensen's Inequality, we get

$$\frac{1}{b}\cdot a\ln a + \frac{1}{c}\cdot b\ln b + c\ln c \;\geq\; \left(\frac{1}{b}+\frac{1}{c}+1\right)\cdot\ln\frac{\frac{a}{b}+\frac{b}{c}+c}{\frac{1}{b}+\frac{1}{c}+1}\,.$$

Thus, it remains to show that

$$\frac{a}{b}+\frac{b}{c}+c \;\geq\; \frac{1}{b}+\frac{1}{c}+1\,,$$

or, equivalently,

$$\frac{ac}{b}+b+c^2 \;\geq\; \frac{c}{b}+1+c\,.$$

Since $ac \geq \frac{1}{b}$, it suffices to show that

$$\frac{1}{b^2}+b+c^2 \;\geq\; \frac{c}{b}+1+c\,.$$

The last inequality can be written as

$$\left(2c-1-\tfrac{1}{b}\right)^2 + \left(1-\tfrac{1}{b}\right)^2(4b+3) \;\geq\; 0\,,$$

which is clearly true. This completes the proof.

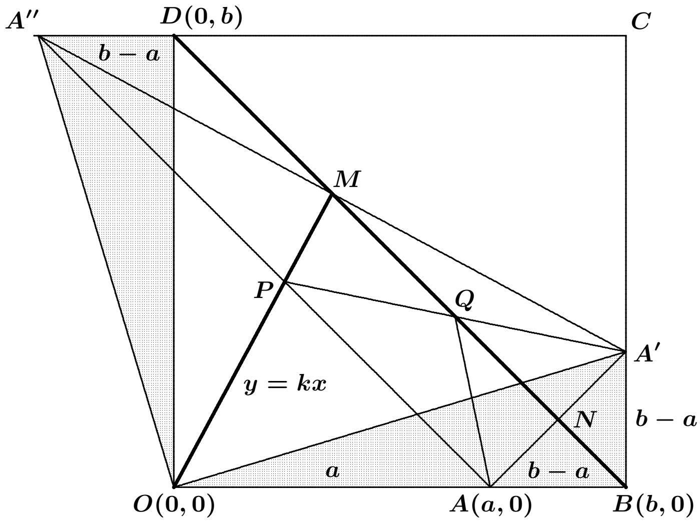
Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA (*part* (a) *only*).

3046. [2005 : 238, 240] *Proposed by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.*

A mirror is placed in the first quadrant of the xy–plane (perpendicular to the plane) along the straight line joining the points $(b,0)$ and $(0,b)$, for some $b>0$. Another mirror is placed similarly along the line $y=kx$ where $k>1$. A light source at $(a,0)$, $0<a<b$, shoots a beam of light into the first quadrant parallel to the first mirror.

Find k such that when the beam is reflected exactly once by each mirror, it passes through the original light source at $(a,0)$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.



Let $OBCD$ be the square whose diagonal is the given mirror that joins $B=(b,0)$ to $D=(0,b)$. Let A' be the mirror image across BD of the light source $A=(a,0)$, and let the line through A parallel to BD intersect CD at A'', while BD intersects $A'A''$ at M and AA' at N. Finally, let $P=AA''\cap OM$ and $Q=PA'\cap BD$.

Since $A''D=AB=A'B=b-a$, the right triangles ODA'' and OBA' are congruent. Hence, $OA''=OA'$ and the lines OA'' and OA' are perpendicular. Since AA'' is parallel to BD and $AN=NA'$, then $A''M=MA'$ and $OM\perp A'A''$. Hence, A'' is the reflection of A' in the mirror OM. Thus, with OM as the initial mirror, the beam AA'' is reflected at P from OM towards A', and reflected again at Q by the mirror BD towards A.

Since $A'A''\perp OM$, the slope k of OM equals the negative reciprocal of the slope of $A''A'$; that is,

$$k=-\frac{A''C}{A'C}=\frac{A''D+DC}{BC-BA'}=\frac{(b-a)+b}{b-(b-a)}=\frac{2b-a}{a}=\frac{2b}{a}-1\,.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; Mª JESúS VILLAR RUBIO, Santander, Spain; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Zhou proved that M is on the bisector of angle PAQ. This implies that the perpendicular bisector of OA passes through M and allows one to calculate the slope of OM easily.

3047. [2005 : 238, 241] *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Evaluate $\sum\limits_{k=1}^{n} \sec\left(\dfrac{2k\pi}{2n+1}\right)$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $z = e^{i\theta} = x + iy$. Then

$$\begin{aligned}\cos\big((2n+1)\theta\big) &= \mathfrak{Re}\big(z^{2n+1}\big) = \sum_{k=0}^{n}\binom{2n+1}{2k}x^{2(n-k)+1}(iy)^{2k} \\ &= \sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k}x^{2(n-k)+1}(1-x^2)^k\,,\end{aligned}$$

which we denote as $f(x)$. It is easy to see that

$$f(x) = a_{2n+1}x^{2n+1} + \cdots + a_1 x\,, \qquad (1)$$

where

$$a_1 = (-1)^n\binom{2n+1}{2n} = (-1)^n(2n+1)\,. \qquad (2)$$

On the other hand, since

$$\begin{aligned}\cos\big((2n+1)\theta\big) &= \frac{1}{2}\left(z^{2n+1}+\frac{1}{z^{2n+1}}\right) = 1+\frac{1}{2}\left(z^{2n+1}+\frac{1}{z^{2n+1}}-2\right) \\ &= 1+\frac{1}{2z^{2n+1}}\big(z^{2n+1}-1\big)^2\end{aligned}$$

and

$$\begin{aligned}\big(z^{2n+1}-1\big)^2 &= \prod_{k=0}^{2n}\left(z-e^{\frac{2k\pi i}{2n+1}}\right)^2 = \prod_{k=0}^{2n}\left(z-e^{\frac{2k\pi i}{2n+1}}\right)\left(z-e^{-\frac{2k\pi i}{2n+1}}\right) \\ &= \prod_{k=0}^{2n}\left(z^2+1-2z\cos\frac{2k\pi}{2n+1}\right)\,,\end{aligned}$$

we also have

$$\begin{aligned} f(x) &= 1+\frac{1}{2}\prod_{k=0}^{2n}\left(z+\frac{1}{z}-2\cos\frac{2k\pi}{2n+1}\right) \\ &= 1+2^{2n}\prod_{k=0}^{2n}\left(x-\cos\frac{2k\pi}{2n+1}\right). \qquad (3) \end{aligned}$$

Now we compare the expressions (1) and (3) for $f(x)$. We see from (1) that the constant term in (3) is zero. Therefore,

$$2^{2n}\prod_{k=0}^{2n}\cos\left(\frac{2k\pi}{2n+1}\right) = 1.$$

Comparing the coefficients of x in (1) and (3), and dividing by the product above (which equals 1), we get

$$a_1 = \sum_{k=0}^{2n}\sec\frac{2k\pi}{2n+1} = 1+2\sum_{k=1}^{n}\sec\frac{2k\pi}{2n+1}.$$

Setting this expression for a_1 equal to the expression in (2), we obtain

$$\sum_{k=1}^{n}\sec\frac{2k\pi}{2n+1} = \frac{(-1)^n(2n+1)-1}{2},$$

which equals n if n is even and $-(n+1)$ if n is odd.

Also solved by ARKADY ALT, San Jose, CA, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

The polynomial $f(x)$ that Zhou used in his solution is closely related to $T_{2n+1}(x)-1$, where T_n is the Chebyshev Polynomial of the first kind, defined by $T_n(\cos\theta) = \cos(n\theta)$. See, for example, `http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html`. *Both Alt and the proposer used T_n in their solutions. Howard used a similar polynomial (ascribed to Waring) and provided reference* [1]. *Janous found the sum in* [2].

References

[1] Aron Pinker, A Generator of Trigonometric Identities. *Two-Year College Math J*, **5**:4 (December 1974), 54–55.

[2] A.P. Prudnikov et al., *Sums and Series (Elementary Functions)* [in Russian], Nauka, Moscow 1981. Page 644, item 4.4.6.2.

3048. [2005 : 238, 241] *Proposed by Gabriel Dospinescu, Paris, France.*

Find all polynomials P with integer coefficients which satisfy the property that, for any relatively prime integers a and b, the sequence $\{P(an+b)\}_{n\geq 1}$ contains an infinite number of terms, any two of which are relatively prime.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

First, P cannot be a constant polynomial, as $\{P(an+b)\}_{n\geq 1}$ would not contain infinitely many terms.

Suppose that $P(x) = kx^m$ for some integers $k \neq 0$ and $m > 0$. Let a and b be relatively prime integers. Then $P(an+b) = k(an+b)^m$ is divisible by k for all n. Therefore P does not satisfy the required condition if $|k| > 1$. If $k = 1$, then $\{P(an+b)\}_{n\geq 1} = \{(an+b)^m\}_{n\geq 1}$. By Dirichlet's Theorem, the sequence $\{an+b\}_{n\geq 1}$ contains infinitely many primes, which implies that the sequence $\{(an+b)^m\}_{n\geq 1}$ contains infinitely many terms that are pairwise relatively prime. Thus, the polynomial $P(x) = x^m$ is a solution for any positive integer m. Then the polynomial $P(x) = -x^m$ must be a solution as well.

Now suppose that P has at least two terms. Then $P(x) = x^\ell Q(x)$ for some non-negative integer ℓ, where

$$Q(x) \;=\; a_j x^j + a_{j-1}x^{j-1} + \cdots + a_1 x + a_0\,,$$

with $j \geq 1$ and $a_j a_0 \neq 0$. Choose a prime q such that $q \nmid a_0$. Since Q is a nonconstant polynomial, we can choose a sufficiently large positive integer r such that $|Q(q^r)| > 1$. Then we can choose a prime p such that $p \mid Q(q^r)$. Note that $p \neq q$, since $Q(q^r) \equiv a_0 \pmod{q}$ and $q \nmid a_0$.

Let $a = p$ and $b = q^r$. Then a and b are relatively prime, since $p \neq q$. Moreover,

$$P(an+b) \;=\; P(pn+q^r) \;\equiv\; P(q^r) \;=\; q^{r\ell}Q(q^r) \;\equiv\; 0 \pmod{p}\,.$$

Thus, all terms of $\{P(an+b)\}_{n\geq 1}$ are divisible by p. Then there cannot be an infinite number of relatively prime terms, which means that P does not satisfy the required condition.

We conclude that the only solutions are the polynomials $P(x) = \pm x^m$ for $m \in \mathbb{N}$.

Also solved by LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

3049. [2005 : 239, 241] *Proposed by Óscar Ciaurri, Universidad de La Rioja, Logroño, Spain and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Given the function $f(x) = \dfrac{x^2}{\sqrt{1+x^2}}\, e^{-\arctan x}$,

(a) find the slant asymptote L in the first quadrant, and

(b) find the area in the first quadrant bounded by the graph of $y = f(x)$ and the line L.

Combination of solutions by Michel Bataille, Rouen, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA; adapted by the editor.

(a) For $x > 1$, we have $\frac{\pi}{2} - \arctan x = \int_x^\infty \frac{1}{1+t^2}\,dt$. Using the substitution $t = 1/w$ we get

$$\begin{aligned}\frac{\pi}{2} - \arctan x &= \sum_{n=0}^{\infty}(-1)^n \int_0^{\frac{1}{x}} w^{2n}\,dw \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)x^{2n+1}} = \frac{1}{x} + O\left(\frac{1}{x^3}\right) .\end{aligned}$$

Hence,

$$e^{\frac{\pi}{2}-\arctan x} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{x} + O\left(\frac{1}{x^3}\right)\right)^n = 1 + \frac{1}{x} + \frac{1}{2x^2} + O\left(\frac{1}{x^3}\right) .$$

Therefore,

$$\begin{aligned} f(x) &= e^{-\frac{\pi}{2}} x \left(1+x^{-2}\right)^{-\frac{1}{2}} e^{\frac{\pi}{2}-\arctan x} \\ &= e^{-\frac{\pi}{2}} x \left(1 + O(x^{-2})\right)\left(1 + x^{-1} + O(x^{-2})\right) ,\end{aligned}$$

which implies that the equation for L is $y = e^{-\frac{\pi}{2}}(x+1)$.

(b) Let $\ell(x) = e^{-\frac{\pi}{2}}(x+1)$, so that the equation for L is $y = \ell(x)$. We claim that $\ell(x) > f(x)$ for $x > 0$. To prove this, we note that the inequality $\ell(x) > f(x)$ is equivalent to

$$-\tfrac{\pi}{2} + \ln(x+1) < 2\ln x - \tfrac{1}{2}\ln(1+x^2) - \arctan x .$$

Replacing x by $1/x$ and using the identity $\arctan x + \arctan(1/x) = \frac{\pi}{2}$, we obtain

$$\ln(x+1) > -\tfrac{1}{2}\ln(1+x^2) + \arctan x . \qquad (1)$$

For $x \geq 0$, let

$$\phi(x) = \ln(1+x) + \tfrac{1}{2}\ln(1+x^2) - \arctan x .$$

Then $\phi(0) = 0$ and $\phi'(x) = \dfrac{2x^2}{(x+1)(1+x^2)} > 0$ for $x > 0$. Therefore, $\phi(x) > 0$ for all $x > 0$. This proves (1) and completes the proof of our claim that $\ell(x) > f(x)$ for $x > 0$.

The desired area is $\int_0^\infty \left(\ell(x) - f(x)\right) dx$. We have

$$\int f(x)\,dx = \int \frac{x^2}{\sqrt{1+x^2}} e^{-\arctan x}\,dx = \int e^{-w} \sec w \tan^2 w\,dw ,$$

where $w = \arctan x$. Integrating by parts yields

$$\begin{aligned}\int f(x)\,dx &= \int e^{-w}\sec w\,d(\tan w) - \int e^{-w}\sec w\,dw \\ &= e^{-w}\sec w\tan w - \int \tan w\,d(e^{-w}) - \int e^{-w}\sec w\,dw \\ &= e^{-w}\sec w\tan w + e^{-w}\sec w - \int e^{-w}\sec w\tan^2 w\,dw \\ &= (x+1)\sqrt{1+x^2}e^{-\arctan x} - \int f(x)\,dx\,.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_0^\infty \big(\ell(x) - f(x)\big)\,dx \\ &= \lim_{u\to\infty}\left[e^{-\pi/2}(\tfrac{1}{2}x^2 + x) - \tfrac{1}{2}(1+x)\sqrt{1+x^2}\,e^{-\arctan x}\right]\Big|_0^u \\ &= \lim_{u\to\infty}\left[e^{-\pi/2}(\tfrac{1}{2}u^2 + u) - \tfrac{1}{2}(1+u)\sqrt{1+u^2}\,e^{-\arctan u}\right] + \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{2}\lim_{u\to\infty} e^{-\frac{\pi}{2}}u^2\left[+2u^{-1} - (1+u^{-1})(1+u^{-2}+O(u^{-4}))\right. \\ &\qquad \left.\cdot(1+u^{-1}+\tfrac{1}{2}u^{-2}+O(u^{-3}))\right] \\ &= \tfrac{1}{2} - e^{-\pi/2}\,.\end{aligned}$$

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; and the proposers. There was one incorrect solution.

3050. [2005 : 239, 241] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle with Cevians AX, BY, CZ. Let L, M, N be the mid-points of AX, BY, CZ, respectively. Let AM and AN meet BC at P_1 and P_2, respectively; let BN and BL meet CA at Q_1 and Q_2, respectively; and let CL and CM meet AB at R_1 and R_2, respectively.

Prove that P_1, P_2, Q_1, Q_2, R_1, R_2 lie on a conic.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

We must assume that the Cevians are concurrent. On the other hand, it is not necessary for L, M, and N to be the mid-points of their respective Cevians. We shall prove that if L, M, N are arbitrary points on the Cevians AX, BY, CZ, respectively, and if $AM \cap BC = P_1$, $BN \cap CA = Q_1$, $CL \cap AB = R_1$, $AN \cap BC = P_2$, $BL \cap CA = Q_2$, and $CM \cap AB = R_2$, then P_1, P_2, Q_1, Q_2, R_1, R_2 lie on a conic if and only if the Cevians AX, BY, and CZ are concurrent.

Let the lines R_1Q_2 and BC meet at P (if the lines are parallel, take P at infinity). Note that X and P are harmonic conjugates with respect to B and C (because B and C are diagonal points of the complete quadrangle AR_1LQ_2, while the diagonals AL and R_1Q_2 meet BC at X and P). In the notation of directed distances, this implies that

$$\frac{BP}{PC} = -\frac{CX}{XB}.$$

Similarly, let P_1R_2 meet CA at Q and let Q_1P_2 meet AB at R. Then, using quadrangles BP_1MR_2 and CQ_1NP_2, we have

$$\frac{CQ}{QA} = -\frac{AY}{YC} \quad \text{and} \quad \frac{AR}{RB} = -\frac{BZ}{ZA}.$$

Multiplying the three equations together, we see that

$$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB} = -\frac{CX}{XB}\cdot\frac{BZ}{ZA}\cdot\frac{AY}{YC}.$$

By the theorems of Ceva and Menelaus applied to triangle ABC, the above equation shows that the lines AX, BY, CZ concur if and only if points P, Q, R are collinear. However, by the converse to Pascal's Theorem, the hexagon $P_1P_2Q_1Q_2R_1R_2$ can be inscribed in a conic if and only if P, Q, R are collinear. Therefore, the points P_1, P_2, Q_1, Q_2, R_1, R_2 lie on a conic if and only if the Cevians AX, BY, CZ are concurrent.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

The proposer tacitly assumed the existence of a common point as part of the definition of the Cevians (so that each Cevian is determined by that point and a vertex). All solvers except Woo and Zhao went along with that interpretation. By contrast, every reference book on this editor's shelf defines a cevian as in the featured solution—a segment joining a vertex to a point on the opposite side—and they write the word using a lower-case c.

3051. [2005 : 333, 335] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let x, y, $z \in [0,1)$ such that $x+y+z=1$. Prove that

(a) $\sqrt{\dfrac{x}{x+yz}}+\sqrt{\dfrac{y}{y+zx}}+\sqrt{\dfrac{z}{z+xy}} \leq 3\sqrt{\dfrac{3}{2}}$;

(b) $\dfrac{\sqrt{xyz}}{(1-x)(1-y)(1-z)} \leq \dfrac{3\sqrt{3}}{8}$.

Similar solutions by Arkady Alt, San Jose, CA, USA; and Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

For part (a) we will prove the stronger inequality

$$\sqrt{\frac{x}{x+yz}}+\sqrt{\frac{y}{y+xz}}+\sqrt{\frac{z}{z+xy}} \leq \frac{3\sqrt{3}}{2}. \tag{1}$$

We may assume for both parts (a) and (b) that $x, y, z \in (0,1)$, because the inequalities are trivial if one of x, y, z is zero. Our proof will be based on the following inequality, which is an immediate consequence of the convexity of the sine function on $[0,\pi]$:

$$\sin\theta_1 + \sin\theta_2 + \sin\theta_3 \;\leq\; \frac{3\sqrt{3}}{2} \tag{2}$$

for all $\theta_1, \theta_2, \theta_3 > 0$ such that $\theta_1 + \theta_2 + \theta_3 = \pi$.

Let $a = y+z$, $b = z+x$, and $c = x+y$. Since $x+y+z = 1$, we have $a = 1-x$, $b = 1-y$, $c = 1-z$, and $a+b+c = 2$. The numbers a, b, c satisfy the triangle inequalities $a+b > c$, $b+c > a$, and $c+a > b$. Therefore, a, b, c are the lengths of the sides of a triangle. Let α, β, γ be the angles opposite the sides a, b, c, respectively. Thus, $\alpha+\beta+\gamma = \pi$. Note that the semiperimeter of the triangle is $s = \frac{1}{2}(a+b+c) = 1$.

(a) We have

$$\begin{aligned}
\frac{x}{x+yz} &= \frac{1-a}{1-a+(1-b)(1-c)} = \frac{1-a}{2-(a+b+c)+bc} \\
&= \frac{1-a}{bc} = \frac{s(s-a)}{bc} = \cos^2(\alpha/2)\,.
\end{aligned}$$

Similarly, $\dfrac{y}{y+zx} = \cos^2(\beta/2)$ and $\dfrac{z}{z+xy} = \cos^2(\gamma/2)$. Thus, the left side of (1) is equal to $\cos(\alpha/2)+\cos(\beta/2)+\cos(\gamma/2)$. To prove (1), it will be sufficient to prove that

$$\cos(\alpha/2)+\cos(\beta/2)+\cos(\gamma/2) \;\leq\; \frac{3\sqrt{3}}{2}$$

for all $\alpha, \beta, \gamma > 0$ such that $\alpha+\beta+\gamma = \pi$. But this follows by applying (2) with $\theta_1 = \frac{1}{2}(\pi-\alpha)$, $\theta_2 = \frac{1}{2}(\pi-\beta)$, and $\theta_3 = \frac{1}{2}(\pi-\gamma)$.

(b) Let K and R be the area and circumradius, respectively, of the triangle with sides a, b, c. Then

$$\begin{aligned}
\frac{\sqrt{xyz}}{(1-x)(1-y)(1-z)} &= \frac{\sqrt{(s-a)(s-b)(s-c)}}{abc} = \frac{sK}{4KR} = \frac{s}{4R} \\
&= \frac{1}{4}\left(\frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}\right) \\
&= \frac{1}{4}(\sin\alpha+\sin\beta+\sin\gamma) \;\leq\; \frac{3\sqrt{3}}{8}\,,
\end{aligned}$$

where the last step follows by applying (2) with $\theta_1 = \alpha$, $\theta_2 = \beta$, and $\theta_3 = \gamma$.

Also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RONGZHENG JIAO, Yangzhou University, Yangzhou, China; PHAM VAN THUAN, Hanoi University of Science, Hanoi, Vietnam; PANOS E. TSAOUSSOGLOU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

All solvers proved the stronger inequality for (a) *which appears in the featured solution.*

3052. [2005 : 333, 335] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Let G be the centroid of $\triangle ABC$, and let A_1, B_1, C_1 be the mid-points of BC, CA, AB, respectively. If P is an arbitrary point in the plane of $\triangle ABC$, show that

$$PA + PB + PC + 3PG \;\geq\; 2(PA_1 + PB_1 + PC_1)\,.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

There is no need to restrict P to the plane of $\triangle ABC$. Indeed, let P be any point of an n–dimensional space that contains $\triangle ABC$. Let $\mathrm{a} = \overrightarrow{PA}$, $\mathrm{b} = \overrightarrow{PB}$, and $\mathrm{c} = \overrightarrow{PC}$. Then $2\overrightarrow{PA_1} = \mathrm{b}+\mathrm{c}$, $2\overrightarrow{PB_1} = \mathrm{c}+\mathrm{a}$, $2\overrightarrow{PC_1} = \mathrm{a}+\mathrm{b}$, and $3\overrightarrow{PG} = \mathrm{a}+\mathrm{b}+\mathrm{c}$. Thus, the inequality under consideration reads

$$|\mathrm{a}| + |\mathrm{b}| + |\mathrm{c}| + |\mathrm{a}+\mathrm{b}+\mathrm{c}| \;\geq\; |\mathrm{a}+\mathrm{b}| + |\mathrm{b}+\mathrm{c}| + |\mathrm{c}+\mathrm{a}|\,.$$

This is the inequality of Hlawka and Hornich. For its proof and for very many extensions of the inequality to various spaces, see D.S. Mitrinović, *Analytic Inequalities*, Springer, Berlin, 1970, pages 170–176; or the more recent D.S. Mitrinović et al., *Classical and New Inequalities in Analysis* (Kluwer Academic Publishers, Dordrecht, 1993), pages 521–534.

The actual inequality from the problem statement can be found in D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989, page 410. The reference there is to the 1984 paper by the Romanian mathematicians M. Chiriţâ and R. Constantinescu.

Also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer. There was one incorrect submission.

Bataille referred to Problem 2482 [1999 : 430; 2000 : 506]*, where further comments and references can be found. Zhao derived the inequality from Popoviciu's Inequality, referring to the* ***CRUX with MAYHEM*** *article "Two Generalizations of Popoviciu's Inequality" by Vasile Cîrtoaje* [2005 : 313–318]*. Both Bencze and Janous provided natural generalizations to k points in n–dimensional space.*

3053. [2005 : 333, 335] *Proposed by Avet A. Grigoryan and Hayk N. Sedrakyan, students, A. Shahinyan Physics and Mathematics School, Yerevan, Armenia.*

Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers whose sum is 1. Prove that

$$n-1 \;\leq\; \sqrt{\frac{1-a_1}{1+a_1}} + \sqrt{\frac{1-a_2}{1+a_2}} + \cdots + \sqrt{\frac{1-a_n}{1+a_n}} \;\leq\; n-2+\frac{2}{\sqrt{3}}\,.$$

Solution by the proposers, modified and expanded by the editor.

Let S_n denote the given summation. Note that, for $0 \le x \le 1$, we have $1 \ge 1 - x^2$, which implies that $\frac{1-x}{1+x} \ge (1-x)^2$; hence, $\sqrt{\frac{1-x}{1+x}} \ge 1 - x$. Now,

$$S_n \;=\; \sum_{k=1}^{n} \sqrt{\frac{1-a_k}{1+a_k}} \;\ge\; \sum_{k=1}^{n}(1-a_k) \;=\; n - \sum_{k=1}^{n} a_k \;=\; n-1\,.$$

This proves the left inequality.

To prove the right inequality, we first establish two lemmas.

Lemma 1. If $0 \le x,\, y \le 1$ such that $x + y = 1$, then

$$\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-y}{1+y}} \;\le\; \frac{2}{\sqrt{3}}\,.$$

Proof: We have

$$\left(\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-y}{1+y}}\right)^2 \;=\; \frac{1-x}{1+x} + \frac{1-y}{1+y} + 2\sqrt{\frac{(1-x)(1-y)}{(1+x)(1+y)}}\,. \qquad (1)$$

Since

$$\frac{1-x}{1+x} + \frac{1-y}{1+y} \;=\; \frac{2(1-xy)}{(1+x)(1+y)} \;=\; \frac{2(1-xy)}{1+x+y+xy} \;=\; \frac{2(1-xy)}{2+xy}$$

$$\text{and}\quad \frac{(1-x)(1-y)}{(1+x)(1+y)} \;=\; \frac{1-(x+y)+xy}{1+x+y+xy} \;=\; \frac{xy}{2+xy}\,,$$

we have, from (1),

$$\left(\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-y}{1+y}}\right)^2 \;=\; \frac{2}{2+xy}\left(1 - xy + \sqrt{xy(2+xy)}\right)\,. \qquad (2)$$

By the AM–GM Inequality, we have

$$\begin{aligned}\sqrt{xy(2+xy)} \;&=\; \tfrac{1}{3}\sqrt{9xy(2+xy)} \\ &\le\; \tfrac{1}{6}(9xy+2+xy) \;=\; \tfrac{1}{3}(1+5xy)\,. \qquad (3)\end{aligned}$$

Substituting (3) into (2), we then obtain

$$\left(\sqrt{\frac{1-x}{1+x}} + \sqrt{\frac{1-y}{1+y}}\right)^2 \;=\; \frac{2}{2+xy}\left(1 - xy + \frac{1+5xy}{3}\right) \;=\; \frac{4}{3}\,,$$

from which the result follows immediately. ∎

Lemma 2. If $x,\, y \ge 0$ such that $x + y \le \frac{4}{5}$, then

$$\frac{1-x}{1+x} + \frac{1-y}{1+y} \;\le\; 1 + \sqrt{\frac{1-x-y}{1+x+y}}\,.$$

Proof: If $x = 0$ or $y = 0$, then clearly equality holds. Suppose $xy \neq 0$. By squaring and rearranging, we obtain the equivalent inequality

$$\begin{aligned} 2&\left(\sqrt{\frac{1-x}{1+x}} \cdot \sqrt{\frac{1-y}{1+y}} - \sqrt{\frac{1-x-y}{1+x+y}}\right) \\ &\leq 1 + \frac{1-x-y}{1+x+y} - \left(\frac{1-x}{1+x} + \frac{1-y}{1+y}\right) \\ &= \frac{2}{1+x+y} - \frac{2(1-xy)}{(1+x)(1+y)} \\ &= \frac{2(1+x+y+xy) - 2(1+x+y-xy-xy(x+y))}{(1+x+y)(1+x)(1+y)} \\ &= \frac{4xy+2xy(x+y)}{(1+x+y)(1+x)(1+y)} = \frac{2xy(2+x+y)}{(1+x+y)(1+x)(1+y)}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\frac{1-x}{1+x} \cdot \frac{1-y}{1+y} - \frac{1-x-y}{1+x+y} \\ &\leq \frac{xy(2+x+y)}{(1+x+y)(1+x)(1+y)} \left(\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}} + \sqrt{\frac{1-x-y}{1+x+y}}\right). \end{aligned} \tag{4}$$

Now,

$$\begin{aligned} &\frac{1-x}{1+x} \cdot \frac{1-y}{1+y} - \frac{1-x-y}{1+x+y} \\ &= \frac{(1-x-y+xy)(1+x+y) - (1-x-y)(1+x+y+xy)}{(1+x+y)(1+x)(1+y)} \\ &= \frac{xy(1+x+y) - (1-x-y)xy}{(1+x+y)(1+x)(1+y)} = \frac{2xy(x+y)}{(1+x+y)(1+x)(1+y)}. \end{aligned}$$

Hence, (4) is equivalent to

$$\frac{2(x+y)}{2+x+y} \leq \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}} + \sqrt{\frac{1-x-y}{1+x+y}}. \tag{5}$$

To prove (5), we note that

$$\begin{aligned} \frac{(1-x)(1-y)}{(1+x)(1+y)} &= \frac{1-x-y+xy}{1+x+y+xy} \geq \frac{1-x-y}{1+x+y} \\ &= -1 + \frac{2}{1+x+y} \geq -1 + \frac{2}{1+(4/5)} = \frac{1}{9}. \end{aligned}$$

Thus,

$$\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}} \geq \sqrt{\frac{1-x-y}{1+x+y}} \geq \frac{1}{3}.$$

Hence,

$$\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}} + \sqrt{\frac{1-x-y}{1+x+y}} \geq 2\sqrt{\frac{1-x-y}{1+x+y}} \geq \frac{2}{3}. \tag{6}$$

We also have

$$\frac{2}{3} > \frac{2(x+y)}{2+x+y}, \tag{7}$$

since $2(2+x+y) - 6(x+y) = 4 - 4(x+y) > 0$. Using (6) and (7), we obtain (5), completing the proof of Lemma 2. ∎

Now we proceed to prove the original right inequality by induction. The case $n = 1$ is trivial, since $a_1 = 1$ implies $S_1 = 0 < -1 + \frac{2}{\sqrt{3}}$. The case $n = 2$ is Lemma 1.

Suppose that, for some $n \geq 3$, we have $S_{n-1} \leq n - 3 + \frac{2}{\sqrt{3}}$ for all non-negative real numbers $a_1, a_2, \ldots, a_n$ that sum to 1. Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers with a sum of 1. Without loss of generality, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Then

$$2 = (a_1 + a_2) + (a_2 + a_3) + \cdots + (a_n + a_1) \geq n(a_1 + a_2).$$

Thus, $a_1 + a_2 \leq \frac{2}{n} \leq \frac{2}{3} < \frac{4}{5}$. Since $(a_1 + a_2) + \sum\limits_{k=3}^{n} a_k = 1$, we have, by Lemma 2 and the induction hypothesis,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \sqrt{\frac{1-a_k}{1+a_k}} \leq 1 + \sqrt{\frac{1-a_1-a_2}{1+a_1+a_2}} + \sum_{k=3}^{n} \sqrt{\frac{1-a_k}{1+a_k}} \\ &\leq 1 + (n-3) + \frac{2}{\sqrt{3}} = n - 2 + \frac{2}{\sqrt{3}}, \end{aligned}$$

completing the induction.

Finally, note that equality holds in both inequalities if and only if either $a_1 = a_2 = \cdots = a_{n-1} = 0$ and $a_n = 1$, or $a_1 = a_2 = \cdots = a_{n-2} = 0$ and $a_{n-1} = a_n = \frac{1}{2}$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and PETER Y. WOO, Biola University, La Mirada, CA, USA. MIHÁLY BENCZE, Brasov, Romania sent in some remarks regarding various upper and lower bounds for S_n under additional assumptions on the quantities a_k. There were also three additional solutions, all of which contained some flaws.

3054. [2005 : 333, 336] *Proposed by Michel Bataille, Rouen, France.*

For $n = 0, 1, 2, \ldots$, let $U_n = \sum\limits_{k=0}^{n} \binom{2k}{k}$ and $V_n = \sum\limits_{k=0}^{n} (-1)^k \binom{2k}{k}$. Evaluate the following in closed form:

(a) $U_n^2 + 2\sum\limits_{k=1}^{n} \binom{2n+2k}{n+k} U_{n-k}$; (b) $V_n^2 + 2\sum\limits_{k=1}^{n} (-1)^{n+k} \binom{2n+2k}{n+k} V_{n-k}$.

Solution by Tom Leong, Brooklyn, NY, USA.

Denote the given expressions in (a) and (b) by A_n and B_n, respectively. It is well known that the generating functions for the sequences $\binom{2k}{k}$ and $(-1)^k\binom{2k}{k}$, for $k = 0, 1, 2, \ldots$, are $\frac{1}{\sqrt{1-4x}}$ and $\frac{1}{\sqrt{1+4x}}$, respectively; that is,

$$\frac{1}{\sqrt{1-4x}} = \sum_{k=0}^{\infty}\binom{2k}{k}x^k \qquad \text{and} \qquad \frac{1}{\sqrt{1+4x}} = \sum_{k=0}^{\infty}(-1)^k\binom{2k}{k}x^k .$$

[See, for example, C.L. Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill, Inc., 1968.]

(a) We claim that

$$\begin{aligned} A_n &= \sum_{k=0}^{2n}\left(\sum_{i+j=k}\binom{2i}{i}\binom{2j}{j}\right) \\ &= \sum_{i+j=0}\binom{2i}{i}\binom{2j}{j} + \sum_{i+j=1}\binom{2i}{i}\binom{2j}{j} + \cdots + \sum_{i+j=2n}\binom{2i}{i}\binom{2j}{j}, \qquad (1) \end{aligned}$$

where, for each $k = 0, 1, 2, \ldots, 2n$, the summation $\sum\limits_{i+j=k}\binom{2i}{i}\binom{2j}{j}$ is over *all* ordered pairs (i,j) of non-negative integers such that $i+j=k$.

Indeed, the terms $\binom{2i}{i}\binom{2j}{j}$ when $i+j \le n$ are all the terms in the expansion of U_n^2. On the other hand, if either $i \ge n+1$ or $j \ge n+1$, say $i = n+k$ for some k with $1 \le k \le n$, then

$$\binom{2i}{i}\binom{2j}{j} = \binom{2n+2k}{n+k}\binom{2j}{j}.$$

These are all the terms in the expansion of $\binom{2n+2k}{n+k}U_{n-k}$. Hence, (1) is established.

Now we recognize that the terms on the right side of (1) are precisely the coefficients of $x^0, x^1, x^2, \ldots, x^{2n}$, respectively, in the product of $\sum\limits_{k=0}^{\infty}\binom{2k}{k}x^k$ with itself. Since

$$\frac{1}{\sqrt{1-4x}}\cdot\frac{1}{\sqrt{1-4x}} = \frac{1}{1-4x} = \sum_{k=0}^{\infty}(4x)^2 ,$$

we conclude that $A_n = 1 + 4 + 4^2 + \cdots + 4^{2n} = \frac{1}{3}\left(4^{2n+1}-1\right)$.

(b) Using a similar argument, we can show that

$$B_n = \sum_{i+j=0} (-1)^0 \binom{2i}{i}\binom{2j}{j} + \sum_{i+j=1} (-1)^1 \binom{2i}{i}\binom{2j}{j} + \cdots + \sum_{i+j=2n} (-1)^{2n} \binom{2i}{i}\binom{2j}{j}, \qquad (2)$$

and that the coefficients on the right side of (2) are precisely the coefficients of x^0, x^1, x^2, ..., x^{2n} in the product of $\sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} x^k$ with itself. Since

$$\frac{1}{\sqrt{1+4x}} \cdot \frac{1}{\sqrt{1+4x}} = \frac{1}{1+4x} = \sum_{k=0}^{\infty} (-1)^k (4x)^k ,$$

we conclude that $B_n = 1 - 4 + 4^2 - \cdots + 4^{2n} = \frac{1}{5}\big(4^{2n+1}+1\big)$.

Also solved by the proposer.

3055. [2005 : 334, 336] *Proposed by Michel Bataille, Rouen, France.*

Let the incircle of an acute-angled triangle ABC be tangent to BC, CA, AB at D, E, F, respectively. Let D_0 be the reflection of D through the incentre of $\triangle ABC$, and let D_1 and D_2 be the reflections of D across the diameters of the incircle through E and F. Define E_0, E_1, E_2 and F_0, F_1, F_2 analogously. Show that

$$[D_0D_1D_2] + [E_0E_1E_2] + [F_0F_1F_2] = [DD_1D_2] = [EE_1E_2] = [FF_1F_2] \le \tfrac{1}{4}[ABC] ,$$

where $[XYZ]$ denotes the area of $\triangle XYZ$.

Solution by the proposer.

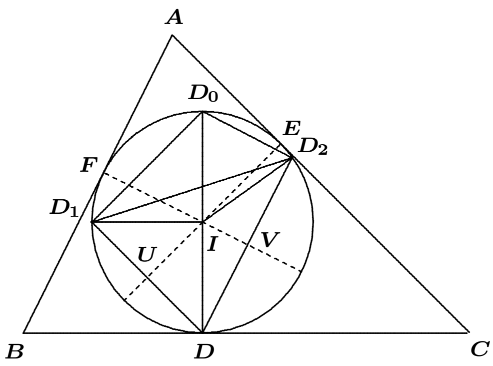
Let I be the incentre of triangle ABC, and let U and V be the midpoints of DD_1 and DD_2, respectively. Since $\angle IEA = \angle IFA = 90°$, we have $\angle UIV = \angle EIF = 180° - A$, and thus, we have

$$\angle D_1DD_2 = \angle UDV = A .$$

Similarly, $\angle EID = 180° - C$, so that $\angle UID = C$. It follows that $\angle DID_1 = 2C$, and that $\angle DD_2D_1 = C$. Consequently, triangles ABC and DD_1D_2 are similar.

The ratio of similarity is the ratio of their circumradii, so that

$$[DD_1D_2] = \left(\frac{r}{R}\right)^2 [ABC] .$$



Analogously, we obtain

$$[EE_1E_2] \;=\; [FF_1F_2] \;=\; \left(\frac{r}{R}\right)^2 [ABC]\,.$$

Now, observe that $\angle D_1D_0D = \angle D_1D_2D = C$, from which it follows that $D_0D_1 = 2r\cos C$. Similarly, we have that $D_0D_2 = 2r\cos B$. We now deduce that

$$[D_0D_1D_2] \;=\; \tfrac{1}{2}\left(4r^2\cos B\cos C\right)\sin A \;=\; \tfrac{1}{2}r^2(\sin 2B + \sin 2C - \sin 2A)\,.$$

Similar relations hold for $[E_0E_1E_2]$ and $[F_0F_1F_2]$. It follows that

$$\begin{aligned} &[D_0D_1D_2] + [E_0E_1E_2] + [F_0F_1F_2] \\ &\quad=\; \frac{1}{2}(\sin 2B + \sin 2C + \sin 2A) \;=\; 2r^2\sin A\sin B\sin C \\ &\quad=\; 2r^2\left(\frac{a}{2R}\right)\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right) \;=\; \frac{r^2}{R^2}\,[ABC]\,. \qquad (1) \end{aligned}$$

This completes the proof of the equalities. The known result $R \ge 2r$ provides the desired inequality.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON.

All the solutions were very similar. This editor chose the proposer's solution because he proved equality (1) first. That alone would have been a very interesting problem! However, Janous did comment "A marvellous problem".

3056. [2005 : 334, 336; 2006 : 44, 47] *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA.*

If $f(x)$ is a non-negative, continuous, concave function on the closed interval $[0,1]$ such that $f(0) = 1$, show that

$$2\int_0^1 x^2 f(x)\,dx + \frac{1}{12} \;\le\; \left[\int_0^1 f(x)\,dx\right]^2\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

More generally, let p be a positive real number and f a continuous and concave function on [0, 1]. We define

$$M_p \;=\; \int_0^1 x^p f(x)\,dx \qquad \text{and} \qquad M_0 \;=\; \int_0^1 f(x)\,dx\,,$$

and show that

$$\frac{p+2}{2}M_p + \frac{2pf(0) - (p+1)}{4(p+1)} \;\le\; M_0^2\,,$$

with equality if and only if $f(x) = m(x - \frac{1}{2}) + \frac{1}{2}$ for some $m \in \mathbb{R}$.

Integrating by parts, we get

$$\begin{aligned} M_p &= \left[x^p \int_0^x f(t)\,dt\right]_0^1 - \int_0^1 \left[px^{p-1} \int_0^x f(t)\,dt\right]\,dx \\ &= M_0 - p\int_0^1\!\!\int_0^x x^{p-1} f(t)\,dt\,dx\,. \end{aligned}$$

Since f is concave on $[0,\ 1]$, we have $f(t) \geq \dfrac{f(x) - f(0)}{x - 0}\,t + f(0)$ for $0 \leq t \leq x \leq 1$. Hence,

$$\begin{aligned} M_p &\leq M_0 - p\int_0^1\!\!\int_0^x \left(x^{p-2}[f(x) - f(0)]\,t + f(0)x^{p-1}\right)\,dt\,dx \\ &= M_0 - \frac{p}{2}\int_0^1 x^p\,[f(x) + f(0)]\,dx \\ &= M_0 - \frac{p}{2}M_p - \frac{pf(0)}{2(p+1)}\,. \end{aligned}$$

Therefore,

$$\frac{p+2}{2}M_p + \frac{2pf(0) - (p+1)}{4(p+1)} \;\leq\; M_0 - \frac{1}{4} \;=\; M_0^2 - \left(M_0 - \frac{1}{2}\right)^2 \;\leq\; M_0^2\,.$$

Inspecting the proof, we see that equality holds if and only if f is a linear function and $M_0 = \frac{1}{2}$. A short calculation shows that this is true if and only if $f(x) = m(x - \frac{1}{2}) + \frac{1}{2}$ for some $m \in \mathbb{R}$.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer. There was one incorrect submission.

All solvers but one used essentially the same argument, but only Zhou replaced 2 by p in the inequality. In particular, they proved (as in our featured solution with $p = 2$) that

$$2\int_0^1 x^2 f(x)\,dx \;\leq\; \int_0^1 f(x)\,dx - \frac{1}{3}\,,$$

which, as Bataille points out, is stronger and more natural. Most mentioned that the problem has also appeared as problem 11133 in The American Mathematical Monthly, **112***:2 (February, 2005), page 180, with the same proposer. Indeed, Zhou submitted our featured solution also to the* Monthly.

3057. [2005 : 334, 336] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c be non-negative real numbers, and let $p \geq \dfrac{\ln 3}{\ln 2} - 1$. Prove that

$$\left(\frac{2a}{b+c}\right)^p + \left(\frac{2b}{c+a}\right)^p + \left(\frac{2c}{a+b}\right)^p \;\geq\; 3\,.$$

Solution by the proposer, expanded slightly by the editor.

Let $x = \frac{2a}{b+c}$, $y = \frac{2b}{c+a}$, and $z = \frac{2c}{a+b}$. Then $x \geq 0$, $y \geq 0$, $z \geq 0$, and the given inequality becomes

$$x^p + y^p + z^p \;\geq\; 3 \tag{1}$$

under the following additional constraint:

$$\frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} \;=\; 1\,. \tag{2}$$

Let $q = \frac{\ln 3}{\ln 2} - 1 \approx 0.585$. By the Power-Mean Inequality, we have

$$\left(\frac{x^p + y^p + z^p}{3}\right)^{\frac{1}{p}} \;\geq\; \left(\frac{x^q + y^q + z^q}{3}\right)^{\frac{1}{q}}\,.$$

Hence, to prove (1), it suffices to show that

$$x^q + y^q + z^q \;\geq\; 3\,. \tag{3}$$

Without loss of generality, we may assume that $a = \min\{a, b, c\}$. Then $x = \frac{2a}{b+c} \leq 1$ and $yz = \frac{4bc}{(c+a)(a+b)} \geq \frac{4bc}{(2c)(2b)} = 1$. Note that (3) can be obtained by adding the following two inequalities:

$$x^q + 2\left(\frac{2}{x+1}\right)^q \;\geq\; 3\,, \tag{4}$$

$$y^q + z^q \;\geq\; 2\left(\frac{2}{x+1}\right)^q\,. \tag{5}$$

We will prove (4) under the constraint $0 \leq x \leq 1$ and (5) under the constraints $yz \geq 1$ and (2). This will suffice to prove (3).

To prove (4), consider the function

$$f(x) \;=\; x^q + 2\left(\frac{2}{x+1}\right)^q\,,$$

for $0 \leq x \leq 1$. Then

$$f'(x) \;=\; qx^{q-1} - q\left(\frac{2}{x+1}\right)^{q+1}\,.$$

Now, for $0 < x \leq 1$, define

$$g(x) \;=\; (q-1)\ln x - (q+1)\ln\left(\frac{2}{x+1}\right)\,.$$

Then $f'(x)$ and $g(x)$ have the same sign on $(0,1]$. Since

$$g'(x) \;=\; \frac{q-1}{x} + (q+1)\left(\frac{1}{x+1}\right) \;=\; \frac{2qx+q-1}{x(x+1)}\,,$$

we have $g'(x)=0$ for $x=x_0=(1-q)/(2q)<1$. Furthermore, $g'(x)<0$ for $x\in(0,x_0)$, and $g'(x)>0$ for $x\in(x_0,1)$. Hence, g is strictly decreasing on $(0,x_0]$ and strictly increasing on $[x_0,1]$.

Since $g(1)=0$ and $\lim\limits_{x\to 0+} g(x)=+\infty$, it follows that there exists $x_1\in(0,x_0)$ such that $g(x_1)=0$. Furthermore, $g(x)>0$ for $x\in(0,x_1)$, and $g(x)<0$ for $x\in(x_1,1)$. Hence, $f'(x_1)=0$, $f'(x)>0$ for $x\in(0,x_1)$, and $f'(x)<0$ for $x\in(x_1,1)$. Therefore, f is strictly increasing on $[0,x_1]$ and strictly decreasing on $[x_1,1]$.

Since $f(0)=2^{q+1}=2^{\ln 3/\ln 2}=2^{\log_2 3}=3$ and $f(1)=3$, we conclude that $f(x)\geq 3$ on $[0,1]$, establishing (4).

To prove (5), we first note that $\left(\frac{y^q+z^q}{2}\right)^{\frac{1}{q}} \geq \left(\frac{\sqrt{y}+\sqrt{z}}{2}\right)^2$, by the Power-Mean Inequality, since $q>1/2$. Therefore, it suffices to show that, for $yz\geq 1$,

$$\left(\frac{\sqrt{y}+\sqrt{z}}{2}\right)^2 \;\geq\; \frac{2}{x+1}\,. \tag{6}$$

From (2), we obtain

$$\frac{1}{x+2} \;=\; 1-\frac{y+z+4}{(y+2)(z+2)} \;=\; \frac{yz+y+z}{yz+2y+2z+4}\,;$$

whence,

$$x+1 \;=\; \frac{yz+2y+2z+4}{yz+y+z}-1 \;=\; \frac{y+z+4}{yz+y+z}\,.$$

Hence, (6) is equivalent to the following, in succession:

$$\begin{aligned}
(y+z+2\sqrt{yz})(y+z+4) &\;\geq\; 8(yz+y+z)\,,\\
(y+z)^2+2(y+z)(\sqrt{yz}-2)+8\sqrt{yz}-8yz &\;\geq\; 0\,,\\
(y+z-2\sqrt{yz})(y+z+4\sqrt{yz}-4) &\;\geq\; 0\,,\\
(\sqrt{y}-\sqrt{z})^2(y+z+4\sqrt{yz}-4) &\;\geq\; 0\,.
\end{aligned}$$

The last inequality is clearly true, since $yz\geq 1$, and this completes the proof.

Note that equality holds if $a=b=c$. In addition, if $p=q$, then equality holds when one of a, b, or c is zero and the other two are equal.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PHAM VAN THUAN, Hanoi University of Science, Hanoi, Vietnam; and PETER Y. WOO, Biola University, La Mirada, CA, USA. MIHÁLY BENCZE, Brasov, Romania sent in six related open questions.

3058. [2005 : 334, 336] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let A, B, C be the angles of a triangle. Prove that

(a) $\displaystyle \frac{1}{2-\cos A}+\frac{1}{2-\cos B}+\frac{1}{2-\cos C} \;\geq\; 2\,;$

(b) $\displaystyle \frac{1}{5-\cos A}+\frac{1}{5-\cos B}+\frac{1}{5-\cos C} \;\leq\; \frac{2}{3}\,.$

Solution by Michel Bataille, Rouen, France; Joe Howard, Portales, NM, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Vedula N. Murty, Dover, PA, USA.

We use the following well-known identities (see [1], pp. 55–56):

$$\prod_{\text{cyclic}} \cos A \;=\; \frac{s^2-4R^2-4Rr-r^2}{4R^2}\,, \tag{1}$$

$$\sum_{\text{cyclic}} \cos A \;=\; \frac{R+r}{R}\,, \tag{2}$$

$$\sum_{\text{cyclic}} \cos B\cos C \;=\; \frac{r^2+s^2-4R^2}{4R^2}\,, \tag{3}$$

and the best quadratic estimates on s^2 (item 5.9 in [2]):

$$16Rr-5r^2 \;\leq\; s^2 \;\leq\; 4R^2+4Rr+3r^2\,. \tag{4}$$

(a) Multiplying both sides of the given inequality by $\prod\limits_{\text{cyclic}}(2-\cos A)$, we obtain the equivalent inequality

$$2\prod_{\text{cyclic}}\cos A+4\sum_{\text{cyclic}}\cos A \;\geq\; 4+3\sum_{\text{cyclic}}\cos B\cos C\,.$$

Using equations (1), (2), and (3), and simplifying, we obtain

$$s^2 \;\leq\; 4R^2+8Rr-5r^2\,.$$

In the light of inequality (4), it suffices to show that

$$4R^2+4Rr+3r^2 \;\leq\; 4R^2+8Rr-5r^2\,.$$

But this is equivalent to $2r \leq R$, which is the well-known Euler Inequality.

(b) Similarly, we obtain the following equivalent form of the desired inequality:

$$72Rr-9r^2 \;\leq\; 5s^2\,.$$

By inequality (4), it suffices to show that

$$72Rr - 9r^2 \ \leq\ 5(16Rr - 5r^2)\,,$$

which is again equivalent to the Euler Inequality $2r \leq R$.

References

[1] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

[2] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969.

Also solved by MIHÁLY BENCZE, Brasov, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Janous has proven the following more general results:

(a) $\dfrac{1}{\lambda-\cos A}+\dfrac{1}{\lambda-\cos B}+\dfrac{1}{\lambda-\cos C}\geq\mu$, *if* $2\leq\mu<6$ *and* $\lambda=\dfrac{\mu+6}{2\mu}$.

(b) $\dfrac{1}{\lambda-\cos A}+\dfrac{1}{\lambda-\cos B}+\dfrac{1}{\lambda-\cos C}\leq\mu$, *if* $0<\mu\leq\dfrac{2}{3}$ *and* $\lambda=\dfrac{\mu+6}{2\mu}$.