KLAMKIN SOLUTIONS

These are the solutions to the special section of problems appearing in the September 2005 issue and dedicated to the memory of Murray S. Klamkin.

KLAMKIN–01. [2005 : 327, 330] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let x and y be positive real numbers from the interval $\left[0, \frac{1}{2}\right]$. Prove that

$$2 \;\le\; \left(\frac{1-x}{1-y}\right)^{\frac{1}{4}} + \left(\frac{1-y}{1-x}\right)^{\frac{1}{4}} \;\le\; \frac{2}{\left(\sqrt{x}\sqrt{y}+\sqrt{1-x}\sqrt{1-y}\right)^{\frac{1}{2}}}\,.$$

(b)★ Is there a generalization of the above inequality to three or more numbers?

I. *Composite of solutions to* (a) *by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

The left inequality follows from the well-known fact that $t + \frac{1}{t} \ge 2$ for $t > 0$, by taking $t = \left(\frac{1-x}{1-y}\right)^{1/4}$.

To prove the right inequality, we make the substitutions $x = \sin^2\alpha$ and $y = \sin^2\beta$, where $\alpha, \beta \in \left[0, \frac{\pi}{4}\right]$. Then the inequality becomes

$$\sqrt{\frac{\cos\alpha}{\cos\beta}} + \sqrt{\frac{\cos\beta}{\cos\alpha}} \;\le\; \frac{2}{\sqrt{\sin\alpha\sin\beta+\cos\alpha\cos\beta}}\,,$$

which is equivalent to $\dfrac{\cos\alpha+\cos\beta}{\sqrt{\cos\alpha\cos\beta}} \le \dfrac{2}{\sqrt{\cos(\alpha-\beta)}}$, or

$$(\cos\alpha+\cos\beta)^2\cos(\alpha-\beta) \;\le\; 4\cos\alpha\cos\beta \tag{1}$$

For notational convenience, we set $u = \frac{1}{2}(\alpha+\beta)$ and $v = \frac{1}{2}(\alpha-\beta)$. Then

$$\begin{aligned}(\cos\alpha+\cos\beta)^2 &= \big(\cos(u+v)+\cos(u-v)\big)^2\\ &= (2\cos u\cos v)^2 \;=\; 4(1-\sin^2 u)(1-\sin^2 v) \qquad (2)\\ \text{and}\quad \cos(\alpha-\beta) &= \cos(2v) \;=\; 1-2\sin^2 v\,. \qquad (3)\end{aligned}$$

Also,

$$\begin{aligned}\cos\alpha\cos\beta &= \tfrac{1}{2}\big[\cos(\alpha+\beta)+\cos(\alpha-\beta)\big]\\ &= \tfrac{1}{2}\big(\cos(2u)+\cos(2v)\big) \;=\; 1-\sin^2 u-\sin^2 v\,. \qquad (4)\end{aligned}$$

Substituting (2), (3), and (4) into (1), gives

$$(1-\sin^2 u)(1-\sin^2 v)(1-2\sin^2 v) \;\le\; 1-\sin^2 u-\sin^2 v\,,$$

which simplifies to

$$\sin^2 u \sin^2 v \;\le\; 2(1-\sin^2 u)(1-\sin^2 v)\sin^2 v\,. \tag{5}$$

Now we will prove (5). Since α, $\beta \in \left[0,\frac{\pi}{4}\right]$, we have $u \in \left[0,\frac{\pi}{4}\right]$ and $v \in \left[-\frac{\pi}{8},\frac{\pi}{8}\right]$. Then $\sin^2 u < \frac{1}{2}$ and $\sin^2 v \le \sin^2 \frac{\pi}{8} < \frac{1}{2}$; hence,

$$2(1-\sin^2 u)(1-\sin^2 v) \;>\; 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) \;=\; \tfrac{1}{2} \;\ge\; \sin^2 u\,,$$

from which (5) follows.

II. *Solution to* (a) *and* (b) *by Li Zhou, Polk Community College, Winter Haven, FL, USA, expanded slightly by the editor.*

For any integer $n \ge 2$, any real number r, and any real numbers $x_1, x_2, \ldots, x_n \in [0,1)$, let

$$F_r(x_1,x_2,\ldots,x_n) \;=\; \left(\prod_{i=1}^{n}(1-x_i)^r\right)^{-\frac{1}{n}} \cdot \sum_{i=1}^{n}(1-x_i)^r\,,$$

and $$G_r(x_1,x_2,\ldots,x_n) \;=\; n\left(\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n}(1-x_i)\right)^{\frac{1}{n}}\right)^{-r}.$$

We prove that

(i) $F_r(x_1,x_2,\ldots,x_n) \ge n$;

(ii) $F_r(x_1,x_2,\ldots,x_n) \le G_r(x_1,x_2,\ldots,x_n)$ if $r > 0$ and $0 \le x_i \le \dfrac{1}{1+r}$ for all i;

(iii) $F_r(x_1,x_2,\ldots,x_n) \;\ge\; G_r(x_1,x_2,\ldots,x_n)$ if $r \le 0$ or if $r > 0$ and $\dfrac{1}{1+r} \le x_i \le 1$ for all i.

Note that (a) is the special case of (i) and (ii) when $n = 2$ and $r = \frac{1}{2}$.

Proof: Part (i) follows immediately from the AM–GM Inequality. For the other two parts, let $f_r(t) = (1+e^t)^{-r}$. Then

$$\begin{aligned} f_r'(t) &= -r(1+e^t)^{-r-1}e^t \\ \text{and}\quad f_r''(t) &= r(r+1)(1+e^t)^{-r-2}e^{2t} - r(1+e^t)^{-r-1}e^t \\ &= r(1+e^t)^{-r-2}e^t(re^t-1)\,. \end{aligned}$$

Hence, f is convex if $r \le 0$ or if $r > 0$ and $t \ge -\ln r$; and concave if $r > 0$ and $t \le -\ln r$. Now let $t_i = \ln\left(\dfrac{x_i}{1-x_i}\right)$ for $0 < x_i < 1$, $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} \sum_{i=1}^{n} f_r(t_i) &= \sum_{i=1}^{n}\left(1+\frac{x_i}{1-x_i}\right)^{-r} = \sum_{i=1}^{n}(1-x_i)^r \\ &= \left(\prod_{i=1}^{n}(1-x_i)^r\right)^{\frac{1}{n}} \cdot F_r(x_1,x_2,\ldots,x_n) \end{aligned} \tag{6}$$

and

$$\begin{aligned} nf_r\left(\frac{1}{n}\sum_{i=1}^{n} t_i\right) &= n\left(1+\exp\left(\frac{1}{n}\sum_{i=n}^{n}\ln\left(\frac{x_i}{1-x_i}\right)\right)\right)^{-r} \\ &= n\left(1+\prod_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{\frac{1}{n}}\right)^{-r} \\ &= n\left(\frac{\left(\prod_{i=1}^{n}(1-x_i)\right)^{\frac{1}{n}}}{\left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}}+\left(\prod_{i=1}^{n}(1-x_i)\right)^{\frac{1}{n}}}\right)^{r} \\ &= \left(\prod_{i=1}^{n}(1-x_i)^r\right)^{\frac{1}{n}}\cdot G_r(x_1,x_2,\ldots,x_n)\,. \qquad (7) \end{aligned}$$

Furthermore, note that $t_i \geq -\ln r$ is equivalent to $\frac{x_i}{1-x_i} \geq \frac{1}{r}$, or $\frac{1-x_i}{x_i} \leq r$, which in turn is equivalent to $\frac{1}{x_i} \leq r+1$, or $\frac{1}{x_i} \geq r+1$. Hence, f is convex if $r \leq 0$ or if $r > 0$ and $\frac{1}{r+1} \leq x_i < 1$ for all i; and concave if $r > 0$ and $0 \leq x_i \leq \frac{1}{r+1}$.

Using (6) and (7) together with Jensen's Inequality applied to $f_r(t)$ yields (ii) and (iii) for $x_i > 0$. The validity of these inequalities when $x_i = 0$ for any i follows from the continuity of $F_r - G_r$ at 0 in each of its variables.

Part (a) *also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; and the proposer. There was one incomplete solution.*

KLAMKIN–02. [2005 : 327, 330] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Let x, y, z be positive real numbers such that $x+y+z=1$. Prove that

$$xyz\left(1+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \;\geq\; \frac{28}{27}\,.$$

(b)★ Prove or disprove the following generalization involving n positive real numbers x_1, x_2, …, x_n which sum to 1:

$$\left(\prod_{i=1}^{n} x_i\right)\left(1+\sum_{i=1}^{n}\frac{1}{x_i^2}\right) \;\geq\; \frac{n^3+1}{n^n}\,.$$

I. *Solution to* (a) *by Chip Curtis, Missouri Southern State University, Joplin, MO, USA, modified by the editor.*

Using the condition that $x+y+z=1$, we obtain

$$\begin{aligned} xyz\left(1+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) &= xyz+\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z} \\ &= \frac{(xyz)^2+(yz)^2+(zx)^2+(xy)^2}{xyz} \\ &= \frac{(xyz)^2+(x+y+z)^2\left[(yz)^2+(zx)^2+(xy)^2\right]}{xyz(x+y+z)^3}. \end{aligned}$$

Thus, the given inequality is equivalent to

$$27(xyz)^2+27(x+y+z)^2\left[(yz)^2+(zx)^2+(xy)^2\right] \;\geq\; 28xyz(x+y+z)^3\,,$$

or

$$\begin{aligned} 27s_1-30xyzs_2+54(x^3y^3+y^3z^3+z^3x^3)& \\ -28xyz(x^3+y^3+z^3)-60x^2y^2z^2 &\;\geq\; 0\,, \end{aligned} \qquad (1)$$

where

$$\begin{aligned} s_1 &= x^4y^2+x^2y^4+y^4z^2+y^2z^4+z^4x^2+z^2x^4\,, \\ s_2 &= x^2y+xy^2+y^2z+yz^2+z^2x+zx^2\,. \end{aligned}$$

Let S denote the left side of (1), and define T, U, V, and W as follows:

$$\begin{aligned} T &= s_1-2xyz(x^3+y^3+z^3)\,, \\ U &= 2(x^3y^3+y^3z^3+z^3x^3)-xyzs_2\,, \\ V &= s_1-xyzs_2\,, \\ \text{and}\quad W &= s_1-6x^2y^2z^2\,. \end{aligned}$$

Then, by tedious (but straightforward) computations, we determine that $S=14T+27U+3V+10W$; whence, (1) is equivalent to

$$14T+27U+3V+10W \;\geq\; 0\,. \qquad (2)$$

To establish (2), it suffices to show that T, U, V, $W\geq 0$.

Consider a majorization relation among the vectors $(4,1,1)$, $(4,2,0)$, $(3,2,1)$, and $(3,3,0)$ in $\mathbb{R}^3$. Since $(4,1,1)\prec(4,2,0)$, $(3,2,1)\prec(3,3,0)$, and $(3,2,1)\prec(4,2,0)$, we have, by Muirhead's Inequality,

$$\begin{aligned} 2xyz(x^3+y^3+z^3) &\leq s_1\,, \\ xyzs_2 &\leq 2(x^3y^3+y^3z^3+z^3x^3)\,, \\ \text{and}\quad xyzs_2 &\leq s_1\,; \end{aligned}$$

that is, T, U, $V \;\geq\; 0$. Also, by the AM–GM Inequality, we see that $s_1\geq 6x^2y^2z^2$; that is, $W\geq 0$.

Hence, (2) is proven, and our proof is complete.

II. *Solution to* (b) *by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.*

The claim is false for $n \geq 4$.

Let $f(x_1, x_2, \ldots, x_n)$ denote the left side of the given inequality. If we set $x_1 = x_2 = \cdots = x_{n-1} = k$ and $x_n = 1-(n-1)k$, where $0 < k < \frac{1}{n}$, then $\sum_{i=1}^{n} x_i = 1$. Since

$$\begin{aligned} & f\bigl(k, k, \ldots, k, 1-(n-1)k\bigr) \\ &\quad = k^{n-1}(1-(n-1)k)\left(1+\frac{n-1}{k^2}+\frac{1}{\bigl(1-(n-1)k\bigr)^2}\right) \\ &\quad = k^{n-3}(1-(n-1)k)\left(k^2+n-1+\frac{k^2}{\bigl(1-(n-1)k\bigr)^2}\right), \end{aligned}$$

we have $\lim_{k\to 0+} f\bigl(k, k, \ldots, k, 1-(n-1)k\bigr) = 0$.

Also solved by ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC. Part (a) *was solved by ARKADY ALT, San Jose, CA, USA; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There was one incomplete solution.*

KLAMKIN–03. [2005 : 327, 330] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

If a, b, c are positive real numbers, prove that

$$\frac{(a+b+c)^2}{a^2+b^2+c^2}+\frac{1}{2}\left(\frac{a^3+b^3+c^3}{abc}-\frac{a^2+b^2+c^2}{ab+bc+ca}\right) \;\geq\; 4\,.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Since $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$, the given inequality is equivalent to

$$\frac{2(ab+bc+ca)}{a^2+b^2+c^2}+\frac{ab^4+ac^4+bc^4+ba^4+ca^4+cb^4}{2abc(ab+bc+ca)} \;\geq\; 3\,. \qquad (1)$$

By the Cauchy–Schwarz Inequality, we have

$$\begin{aligned} & \left(\frac{1}{a}+\frac{1}{a}+\frac{1}{b}+\frac{1}{b}+\frac{1}{c}+\frac{1}{c}\right)\left(ab^4+ac^4+bc^4+ba^4+ca^4+cb^4\right) \\ &\qquad \geq\; \left(b^2+c^2+c^2+a^2+a^2+b^2\right)^2 \\ &\qquad =\; 4\left(a^2+b^2+c^2\right)^2, \end{aligned}$$

from which we obtain

$$\left(\frac{ab+bc+ca}{abc}\right)\left(ab^4+ac^4+bc^4+ba^4+ca^4+cb^4\right) \geq 2\left(a^2+b^2+c^2\right)^2 . \qquad (2)$$

Using the AM–GM Inequality and (2), we then have

$$\frac{2(ab+bc+ca)}{a^2+b^2+c^2}+\frac{ab^4+ac^4+bc^4+ba^4+ca^4+cb^4}{2abc(ab+bc+ca)}$$
$$\geq 3\left(\frac{(ab+bc+ca)\left(ab^4+ac^4+bc^4+ba^4+ca^4+cb^4\right)}{2abc\left(a^2+b^2+c^2\right)^2}\right)^{\frac{1}{3}} \geq 3 ,$$

and (1) is established.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

KLAMKIN–04. [2005 : 327, 330] *Proposed by Mihály Bencze, Brasov, Romania.*

Let f_n denote the Fibonacci sequence (that is, $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$). For all integers $k \geq 1$, determine the remainder when f_{kn-r} is divided by f_n^2 for the following cases:

(a) $r = 1$; (b) $r = 2$; (c)★ $r \in \{3, 4, \ldots, k-1\}$.

Solution by Robert B. Israel, University of British Columbia, Vancouver, BC.

We will designate the Fibonacci numbers to start at 0; that is, $F_0 = 0$ and $f_j = F_{j+1}$, for $j = 0, 1, \ldots$. One convenient way to obtain them is using the powers of the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. For each positive integer n,

$$M^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_n + F_{n-1} \end{pmatrix} .$$

It is easy to prove by induction on k that

$$M^{kn} \equiv \begin{pmatrix} F_{n-1}^k & kF_{n-1}^{k-1}F_n \\ kF_{n-1}^{k-1}F_n & F_{n-1}^k + kF_{n-1}^{k-1}F_n \end{pmatrix} \pmod{F_n^2}$$

for all integers k and all integers n. Moreover, using $F_{-j} = (-1)^{j+1}F_j$ and $M^{kn-r} = M^{-r}M^{kn}$, we get

$$F_{kn-1} \equiv (-1)^r\left(kF_{r-1}F_{n-1}^{k-1}F_n - F_rF_{n-1}^k\right) \pmod{F_n^2} .$$

In the other notation, after replacing n by $n+1$ and r by $r+k-1$, we obtain

$$f_{kn-r} \equiv (-1)^{r+k}\left(f_{r+k-2}f_{n-1}^{k} - kf_{r+k-3}f_{n-1}^{k-1}f_n\right) \pmod{f_n^2} .$$

Parts (a) *and* (b) *also solved by the proposer.*

Note that the matrix representation of the Fibonacci numbers also occurs in this issue in the solution of 3044 on page 331.

KLAMKIN–05. [2005 : 328, 330] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let k and n be positive integers with $k < n$, and let a_1, a_2, $\dots$, a_n be real numbers such that $a_1 \le a_2 \le \cdots \le a_n$. Prove that

$$(a_1 + a_2 + \cdots + a_n)^2 \ \ge\ n(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_na_{n+k})$$

(where the subscripts are taken modulo n) in the following cases:

(a) $n = 2k$; (b) $n = 4k$; (c)★ $2 < \frac{n}{k} < 4$.

Solution by the proposer.

(a) We have to prove that

$$(a_1 + a_2 + \cdots + a_{2k})^2 \ \ge\ 4k(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_ka_{2k}) .$$

Let x be a real number such that $a_k \le x \le a_{k+1}$. Then, obviously,

$$(x - a_1)(a_{k+1} - x) + (x - a_2)(a_{k+2} - x) + \cdots + (x - a_k)(a_{2k} - x) \ \ge\ 0 .$$

Expanding, rearranging, and multiplying by $4k$, we obtain

$$4kx(a_1+a_2+\cdots+a_{2k}) \ \ge\ 4k^2x^2+4k(a_1a_{k+1}+a_2a_{k+2}+\cdots+a_ka_{2k}) . \quad (1)$$

On the other hand, we have the obvious inequality

$$(a_1 + a_2 + \cdots + a_{2k} - 2kx)^2 \ \ge\ 0 ,$$

which can be written as

$$(a_1 + a_2 + \cdots + a_{2k})^2 + 4k^2x^2 \ \ge\ 4kx(a_1 + a_2 + \cdots + a_{2k}) . \quad (2)$$

Adding inequalities (1) and (2), we obtain the desired inequality.

(b) Let $b_i = a_i + a_{2k+i}$ for each integer i, $1 \le i \le 2k$. Clearly, $b_1 \le b_2 \le \cdots \le b_{2k}$. Applying the inequality from part (a), we obtain

$$(b_1 + b_2 + \cdots + b_{2k})^2 \ \ge\ 4k(b_1b_{k+1} + b_2b_{k+2} + \cdots + b_kb_{2k}) ,$$

which is the desired inequality.

There were no other solutions submitted.

KLAMKIN–06. [2005 : 328, 331] *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let Γ be the circumcircle of $\triangle ABC$.

(a) Suppose that the median and the interior angle bisector from A intersect BC at M and N, respectively. Extend AM and AN to intersect Γ at M' and N', respectively. Prove that $MM' \geq NN'$.

(b)★ Suppose that P is a point in the interior of side BC and AP intersects Γ at P'. Find the location of P where PP' is maximal. Is this maximal P constructible by straightedge and compass?

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain.

(a) First we note that the correct inequality should be $MM' \leq NN'$. We use the following well-known (or easy to prove) relations:

$$CN = \frac{ab}{b+c}, \qquad NB = \frac{ac}{b+c}, \qquad AN = \frac{\sqrt{bc}}{b+c}\sqrt{(b+c)^2-a^2}$$

and

$$AM = \frac{\sqrt{2b^2+2c^2-a^2}}{2}.$$

By the Intersecting Chords Theorem,

$$MM' = \frac{CM\cdot MB}{AM} = \frac{a^2}{2\sqrt{2b^2+2c^2-a^2}}$$

and

$$NN' = \frac{CN\cdot NB}{AN} = \frac{a^2\sqrt{bc}}{(b+c)\sqrt{(b+c)^2-a^2}}.$$

Thus, the inequality $MM' \leq NN'$ is equivalent to

$$\frac{1}{2\sqrt{2b^2+2c^2-a^2}} \leq \frac{\sqrt{bc}}{(b+c)\sqrt{(b+c)^2-a^2}},$$

$$\text{or}\quad (b+c)^2 \leq \frac{4bc(2b^2+2c^2-a^2)}{(b+c)^2-a^2}. \tag{1}$$

To prove (1), we calculate

$$\begin{aligned}
&\frac{4bc(2b^2+2c^2-a^2)}{(b+c)^2-a^2} - (b+c)^2\\
&= \frac{4bc(2b^2+2c^2-a^2)}{(b+c)^2-a^2} - 4bc - (b-c)^2\\
&= \frac{4bc(b-c)^2}{(b+c)^2-a^2} - (b-c)^2 = (b-c)^2\left[\frac{4bc-(b+c)^2+a^2}{(b+c)^2-a^2}\right]\\
&= (b-c)^2\left[\frac{a^2-(b-c)^2}{(b+c)^2-a^2}\right] = (b-c)^2\frac{(a-b+c)(a+b-c)}{(-a+b+c)(a+b+c)}.
\end{aligned}$$

Using the Triangle Inequalities $b+c>a$, $c+a>b$, and $a+b>c$, we see that the last expression above is non-negative, which proves (1). Equality holds if and only if $b=c$.

(b) Let $PB=x$. Clearly, $x\in(0,a)$. Using the Intersecting Chords Theorem and the Cosine Law for $\triangle APB$, we obtain

$$PP' \;=\; \frac{CP\cdot PB}{AP} \;=\; \frac{(a-x)\cdot x}{\sqrt{x^2+c^2-2cx\cos B}}\,.$$

Let

$$L(x) \;=\; \frac{x(a-x)}{\sqrt{x^2+c^2-2cx\cos B}}\,.$$

Consider the function $L(x)$ on the interval $[0,a]$. It is positive on the interval $(0,a)$ and continuous on the interval $[0,a]$. Since $L(0)=L(a)=0$, the function $L(x)$ must have a maximum on the interval $(0,a)$. Let x be a critical point for $L(x)$ on $(0,a)$ such that $L'(x)=0$. Then

$$\begin{aligned} 0 &= \frac{L'(x)}{L(x)} \;=\; [\ln L(x)]' \\ &= \left[\ln x+\ln(a-x)-\tfrac{1}{2}\ln(x^2+c^2-2cx\cos B)\right]' \\ &= \frac{1}{x}-\frac{1}{a-x}-\frac{x-c\cos B}{x^2+c^2-2cx\cos B}\,, \end{aligned}$$

which gives the following equation for the critical point x:

$$x^3-(3c\cos B)x^2+(2c^2+ac\cos B)x-ac^2 \;=\; 0\,.$$

Applying the Cosine Law to $\triangle ABC$, we get $2ac\cos B=a^2+c^2-b^2$, and, simplifying, we can rewrite this cubic equation as

$$2ax^3-3(a^2+c^2-b^2)x^2+a(5c^2+a^2-b^2)x-2a^2c^2 \;=\; 0\,. \qquad (2)$$

Let $Q(x)$ denote the cubic on the left side of the above equation. Note that $Q(0)=-2a^2c^2<0$ and $Q(a)=2a^2b^2>0$, which implies that $Q(x)$ has a root in the interval $(0,a)$. Consider a triangle with sides $a=5$, $b=3$, and $c=4$. Equation (2) becomes

$$5x^3-48x^2+240x-400 \;=\; 0\,.$$

The polynomial $5x^3-48x^2+240x-400$ is irreducible over the field of rationals, implying that its only real root, $x=\frac{1}{5}\left(16-4\sqrt[3]{36}+6\sqrt[3]{6}\right)$, is algebraic of degree 3 over the rationals. Consequently, x is not constructible by straightedge and compass. Hence, our conclusion is: Generally, the point P is not constructible by straightedge and compass.

Part (a) *also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There was one incorrect solution.*

KLAMKIN–07. [2005 : 328, 331] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c, d be real numbers such that $a > b \geq c > d > 0$. If $ad - bc > 0$, prove that

$$\prod_{k=1}^{n} \left(\frac{a^{\binom{n}{k}} - b^{\binom{n}{k}}}{c^{\binom{n}{k}} - d^{\binom{n}{k}}} \right)^{k} \geq \left(\frac{a^{\frac{2^n}{n+1}} - b^{\frac{2^n}{n+1}}}{c^{\frac{2^n}{n+1}} - d^{\frac{2^n}{n+1}}} \right)^{\binom{n+1}{2}} .$$

Combination of the solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

We first set

$$f(x) = \ln\left(\frac{a^x - b^x}{c^x - d^x} \right) ,$$

for $x > 0$. Taking logarithms, we see that the given inequality is equivalent to

$$\sum_{k=1}^{n} k \cdot f\left[\binom{n}{k} \right] \geq \binom{n+1}{2} \cdot f\left(\frac{2^n}{n+1} \right) ;$$

that is,

$$\sum_{k=1}^{n} \frac{2k}{n(n+1)} \cdot f\left[\binom{n}{k} \right] \geq f\left(\frac{2^n}{n+1} \right) ,$$

where we have

$$\sum_{k=1}^{n} \frac{2k}{n(n+1)} = 1 .$$

We claim that $f(x)$ is a convex function, but defer the proof until later. Jensen's Inequality then implies that

$$\sum_{k=1}^{n} \frac{2k}{n(n+1)} \cdot f\left[\binom{n}{k} \right] \geq \sum_{k=1}^{n} f\left(\frac{2k}{n(n+1)} \cdot \binom{n}{k} \right) . \qquad (1)$$

Now, the identity $\sum\limits_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ shows that

$$\sum_{k=1}^{n} \frac{2k}{n(n+1)} \cdot \binom{n}{k} = \frac{2^n}{n+1} .$$

The stated inequality follows from this equation and (1).

It remains to prove that $f(x)$ is (strictly) convex on $(0, +\infty)$. Since

$$f(x) = \ln\left(\frac{\left(\frac{a}{b}\right)^x - 1}{\left(\frac{c}{d}\right)^x - 1} \right) + x \ln\left(\frac{b}{d} \right) ,$$

it is sufficient to show that, if $\alpha > \beta > 1$, then $\ln\left(\dfrac{\alpha^x - 1}{\beta^x - 1}\right)$ is convex. But

$$\ln\left(\frac{\alpha^x - 1}{\beta^x - 1}\right) \;=\; \ln\left(\frac{e^{x\ln\alpha} - 1}{e^{x\ln\beta} - 1}\right),$$

and it is then sufficient to show that, if $\gamma > \delta > 0$, then $g(x) = \ln\left(\frac{e^{\gamma x} - 1}{e^{\delta x} - 1}\right)$ is convex. Taking derivatives, we get

$$g''(x) \;=\; \frac{\delta^2 e^{\delta x}}{(e^{\delta x} - 1)^2} - \frac{\gamma^2 e^{\gamma x}}{(e^{\gamma x} - 1)^2}\,.$$

Then $g''(x) > 0$ if and only if $\dfrac{e^{\gamma x} - 1}{\gamma e^{\frac{1}{2}\gamma x}} > \dfrac{e^{\delta x} - 1}{\delta e^{\frac{1}{2}\delta x}}$. We can rewrite this inequality as

$$\frac{\sinh(\frac{1}{2}\gamma x)}{\frac{1}{2}\gamma x} > \frac{\sinh(\frac{1}{2}\delta x)}{\frac{1}{2}\delta x}\,.$$

Since $\dfrac{\sinh x}{x}$ is strictly increasing on $(0, +\infty)$, the preceding inequality holds, and we are done.

There were no other solutions submitted.

KLAMKIN-08. [2005 : 328, 331] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let m and n be positive integers, and let x_1, x_2, $\ldots$, x_m be positive real numbers. If λ is a real number, $\lambda \geq 1$, prove that

$$\left(\prod_{i=1}^{m} x_i\right)^{\frac{1}{m}} \;\leq\; \left(\frac{\lambda\left(\sum\limits_{i=1}^{m} x_i\right)^n + (1-\lambda)\sum\limits_{i=1}^{m} x_i^n}{\lambda m^n + (1-\lambda)m}\right)^{\frac{1}{n}} \;\leq\; \frac{1}{m}\sum_{i=1}^{m} x_i\,.$$

I. *Essentially the same solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; with some detail added by the editor.*

Lemma. Let m and n be positive integers, and let x_1, x_2, $\ldots$, x_n be positive real numbers. Then

$$\left(\sum_{i=1}^{m} x_i\right)^n - \sum_{i=1}^{m} x_i^n \;\geq\; (m^n - m)\left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}}.$$

[*Ed.* The second solution below gives a reference for this inequality.]

Proof: If we expand $\left(\sum_{i=1}^{m} x_i\right)^n$ completely, without combining like terms, we get

$$\left(\sum_{i=1}^{m} x_i\right)^n = \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

The number of elementary terms $x_{i_1}x_{i_2}\cdots x_{i_n}$ on the right side is m^n. Let p denote the total number of times that a given variable, say x_1, occurs as a factor in these terms. (By symmetry, this number is independent of the choice of variable.) Letting $s = \sum_{i=2}^{m} x_i$, we have

$$\left(\sum_{i=1}^{m} x_i\right)^n = (x_1+s)^n = \sum_{k=0}^{n}\binom{n}{k}x_1^{n-k}s^k,$$

by the Binomial Theorem. Then p is the total number of times that x_1 occurs as a factor in the terms on the right side of this equation. Since the expansion of s^k contains $(m-1)^k$ elementary terms, we have

$$\begin{aligned} p &= \sum_{k=0}^{n-1}\binom{n}{k}(n-k)(m-1)^k = \sum_{k=0}^{n-1} n\binom{n-1}{k}(m-1)^k \\ &= n\big[1+(m-1)\big]^{n-1} = nm^{n-1}. \end{aligned}$$

We now subtract $\sum_{i=1}^{m} x_i^n$ from the expansion of $\left(\sum_{i=1}^{m} x_i\right)^n$. The number of terms remaining in the expansion is then $m^n - m$. Applying the AM–GM Inequality to these terms, and noting that the total number of times each variable occurs as a factor in the terms is $p - n = nm^{m-1} - n$, we get

$$\frac{1}{m^n - m}\left[\left(\sum_{i=1}^{m} x_i\right)^n - \sum_{i=1}^{m} x_i^n\right] \geq \left(\prod_{i=1}^{m} x_i^{nm^{m-1}-n}\right)^{\frac{1}{m^n-m}} = \left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}},$$

which gives the desired result. ■

Now consider the left inequality in the given problem. By the AM–GM Inequality, we have $\sum_{i=1}^{m} x_i^n \geq m\left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}}$. Using this inequality and the lemma, we get

$$\lambda\left[\left(\sum_{i=1}^{m} x_i\right)^n - \sum_{i=1}^{m} x_i^n\right] + \sum_{i=1}^{m} x_i^n \geq \Big(\lambda(m^n-m)+m\Big)\left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}},$$

which can be rearranged to give the left inequality.

For the right inequality, we start with an application of the Power Mean Inequality:

$$\frac{1}{m}\sum_{i=1}^{m} x_i^n \geq \left(\frac{1}{m}\sum_{i=1}^{m} x_i\right)^n.$$

Since $\lambda \geq 1$, this inequality is equivalent to

$$(1-\lambda)\sum_{i=1}^{m} x_i^n \;\leq\; \frac{1-\lambda}{m^{n-1}}\left(\sum_{i=1}^{m} x_i\right)^n .$$

Adding $\lambda\left(\sum\limits_{i=1}^{m} x_i\right)^n$ to both sides, we get

$$\lambda\left(\sum_{i=1}^{m} x_i\right)^n + (1-\lambda)\sum_{i=1}^{m} x_i^n \;\leq\; \frac{\lambda m^n + (1-\lambda)m}{m^n}\left(\sum_{i=1}^{m} x_i\right)^n ,$$

which can be rearranged to give the right inequality.

II. *Solution by the proposer.*

Let $a = \sum\limits_{i=1}^{m} x_i^n$ and $b = \left(\sum\limits_{i=1}^{m} x_i\right)^n$. Define

$$f(\lambda) \;=\; \frac{a+\lambda(b-a)}{m+\lambda(m^n-m)} .$$

Taking n^{th} powers throughout the proposed inequalities, and using the notation we have just introduced, we obtain the equivalent inequalities

$$\left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}} \;\leq\; f(\lambda) \;\leq\; f(1)\,, \tag{2}$$

which are to be proved for all $\lambda \geq 1$.

The derivative of $f(\lambda)$ is

$$f'(\lambda) \;=\; \frac{m(b-am^{n-1})}{\Big(m+\lambda(m^n-m)\Big)^2} .$$

By the Power Mean Inequality, we have $b \leq m^{n-1}a$. Thus, $f'(\lambda) \leq 0$ for all λ. The right inequality in (2) then follows immediately, for all $\lambda \geq 1$. Furthermore, we must have

$$f(\lambda) \;\geq\; \lim_{\lambda\to\infty} f(\lambda) \;=\; \frac{b-a}{m^n-m} . \tag{3}$$

Now we make use of the following inequality, which is inequality (2.8) in reference [1]:

$$\left(\sum_{i=1}^{m} x_i\right)^n - \sum_{i=1}^{m} x_i^n \;\geq\; (m^n-m)\left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}} .$$

[*Ed.* Note that this is the inequality in the lemma in the solution above.]

Rearranging this inequality and using our notation, we obtain

$$\frac{b-a}{m^n - m} \;\geq\; \left(\prod_{i=1}^{m} x_i\right)^{\frac{n}{m}} .$$

Combining this with (3), we obtain the left inequality in (2).

Reference

[1] Leng Gangsong, M. Tongyi and Qian Xiangzhong, *Inequalities for a Simplex and an Interior Point*, Geometriae Dedicata 85, pp. 1–10, 2001, Kluwer Academic Publishers, The Netherlands.

There were no other solutions submitted.

KLAMKIN–09. [2005 : 328, 331] *Proposed by Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA.*

For $0 < x < \pi/2$, prove or disprove that

$$\frac{\ln(1-\sin x)}{\ln(\cos x)} \;<\; \frac{2+x}{x} .$$

Essentially the same solution by Michel Bataille, Rouen, France; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

We prove the claim. Multiplying by the negative number $x\ln(\cos x)$, we see that the proposed inequality is equivalent to $u(x) > 0$, where

$$u(x) \;=\; x\ln(1-\sin x) - (x+2)\ln(\cos x) .$$

The first two derivatives of this function are easily obtained:

$$\begin{aligned} u'(x) &= \ln\left(\frac{1-\sin x}{\cos x}\right) - \frac{x}{\cos x} + 2\tan x \\ u''(x) &= \frac{2(1-\cos x) - x\sin x}{\cos^2 x} \;=\; \frac{2\sin x}{\cos^2 x}\left(\tan\frac{x}{2} - \frac{x}{2}\right) . \end{aligned}$$

Now, since $\tan\alpha > \alpha$ for $\alpha \in (0,\pi/2)$, we have $u''(x) > 0$ for $x \in (0,\pi/2)$; hence, the function u' is increasing. Since $\lim\limits_{x\to 0} u'(x) = 0$, we have $u'(x) > 0$ for $x \in (0,\pi/2)$. Thus, u is increasing and, since $\lim\limits_{x\to 0} u(x) = 0$, we have $u(x) > 0$ for all $x \in (0,\pi/2)$.

Also solved by JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Janous noted that the inequality is very sharp near $x = 0$. Indeed, the Taylor expansion about $x = 0$ of the function $u(x)$ in the solution above is $\frac{1}{360}x^6 + O(x^8)$.

KLAMKIN–10. *Proposed by Mihály Bencze, Brasov, Romania.*

Let $P(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial with real coefficients and simple roots. Prove that

(a) $\displaystyle\sum_{i=1}^{n} \frac{1}{x_i P'(x_i)} = -\frac{1}{a_0}$, and (b) $\displaystyle\sum_{i=1}^{n} \frac{x_i^{n-1}}{P'(x_i)} = \frac{1}{a_n}$.

Solution by Joel Schlosberg, Bayside, NY, USA.

For the expressions to make sense, none of the roots can be zero. Consider the function

$$Q(x) = \sum_i \frac{r_i^k P(x)}{P'(r_i)(x - r_i)} - x^k \,,$$

where $P(x)$ is an n^{th} degree polynomial with simple roots $r_1, r_2, \ldots, r_n$ and k is any integer with $0 \le k \le n-1$. Since $x - r_i \mid P(x)$ for all i, the function $Q(x)$ is a polynomial of degree (at most) $n - 1$. For any root r_j

$$\begin{aligned} Q(r_j) &= \sum_{i \neq j} \frac{r_i^k P(r_j)}{P'(r_i)(r_j - r_i)} + \lim_{x \to r_j} \frac{r_j^k P(x)}{P'(r_j)(x - r_j)} - r_j^k \\ &= \lim_{x \to r_j} \frac{r_j^k}{P'(r_j)} \frac{P(x) - P(r_j)}{x - r_j} - r_j^k = \frac{r_j^k P'(r_j)}{P'(r_j)} - r_j^k = 0 \,. \end{aligned}$$

Therefore, $Q(x) = 0$ for n distinct values $r_1, r_2, \ldots, r_n$, but since $Q(x)$ has degree at most $n - 1$, it must be identically zero; thus,

$$\sum_i \frac{r_i^k P(x)}{P'(r_i)(x - r_i)} - x^k = 0 \,,$$

and, if $x \neq r_i$ for all i, we have

$$\sum_i \frac{r_i^k}{P'(r_i)(x - r_i)} = \frac{x^k}{P(x)} \,. \tag{1}$$

For $k = 0$ in (1), we have

$$\sum_i \frac{1}{P'(r_i)(x - r_i)} = \frac{1}{P(x)} \,,$$

and, by setting $x = 0$, (a) is proved, since

$$\sum_i \frac{1}{P'(r_i) r_i} = -\frac{1}{P(0)} = -\frac{1}{a_0} \,.$$

For $k = n-1$ in (1), we have

$$\sum_i \frac{r_i^{n-1}}{P'(r_i)(x-r_i)} = \frac{x^{n-1}}{P(x)} \quad \text{and} \quad \sum_i \frac{r_i^{n-1}}{P'(r_i)} \frac{x}{x-r_i} = \frac{x^n}{P(x)}.$$

By taking the infinite limit, (b) is proved, since

$$\lim_{x\to\infty} \sum_i \frac{r_i^{n-1}}{P'(r_i)} \frac{x}{x-r_i} = \lim_{x\to\infty} \frac{x^n}{P(x)} \quad \text{and} \quad \sum_i \frac{r_i^{n-1}}{P'(r_i)} = \frac{1}{a_n}.$$

Additionally, for $0 \le k \le n-2$ (that is, $1 \le k+1 \le n-1$) in (1), we see that

$$\sum_i \frac{r_i^{k+1}}{P'(r_i)(x-r_i)} = \frac{x^{k+1}}{P(x)},$$

and for $x = 0$, we have $-\sum_i \frac{r_i^{k+1}}{P'(r_i)(r_i)} = 0$ and $\sum_i \frac{r_i^k}{P'(r_i)} = 0$. We conclude that

$$\sum_i \frac{r_i^k}{P'(r_i)} = \begin{cases} -\frac{1}{a_0} & \text{if } k = -1, \\ 0 & \text{if } 0 \le k \le n-2, \\ \frac{1}{a_n} & \text{if } k = n-1. \end{cases}$$

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; VEDULA N. MURTY, Dover, PA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

KLAMKIN–11. [2005 : 329, 332] *Proposed by Mohammed Aassila, Strasbourg, France.*

Let P be an interior point of a triangle ABC, and let r_1, r_2, and r_3 be the inradii of the triangles APB, BPC, and CPA, respectively. Prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \ge \frac{6+4\sqrt{3}}{R},$$

where R is the circumradius of triangle ABC. When does equality hold?

Solution by Scott Brown, Auburn University, Montgomery, AL, USA.

According to [1], we have the following:

$$\frac{r}{r_1} = 1 + \frac{1}{\sin C}, \qquad \frac{r}{r_2} = 1 + \frac{1}{\sin A}, \qquad \frac{r}{r_3} = 1 + \frac{1}{\sin B},$$

where r_1, r_2, and r_3 are the inradii of triangles APB, BPC, and CPA, respectively. Hence,

$$\begin{aligned}\frac{r}{r_1}+\frac{r}{r_2}+\frac{r}{r_3} &= 3+\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)\\ &\geq 3+3\cdot\sqrt[3]{\frac{1}{\sin A\sin B\sin C}}\,,\end{aligned}$$

by the AM–GM Inequality. From [2], we have $\sin A\sin B\sin C \leq \frac{3\sqrt{3}}{8}$. It follows that

$$\frac{r}{r_1}+\frac{r}{r_2}+\frac{r}{r_3} \geq 3+3\left(\sqrt[3]{\frac{8}{3\sqrt{3}}}\right) = 3+2\sqrt{3}\,.$$

Since $R \geq 2r$ (see [2]), we have

$$\frac{R}{r_1}+\frac{R}{r_2}+\frac{R}{r_3} \geq 2\left(\frac{r}{r_1}+\frac{r}{r_2}+\frac{r}{r_3}\right) \geq 6+4\sqrt{3}\,.$$

The desired result follows.

Equality occurs when $\triangle ABC$ is equilateral.

References

[1] Matematika Skole, problem 378, pp. 75–76, No. 1, 1968.

[2] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović & P.M. Vasić, *Geometric Inequalities*, Groningen, 1969.

Also solved by MIHÁLY BENCZE, Brasov, Romania; and the proposer.

KLAMKIN–12. [2005 : 329, 332] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, c be the sides of a spherical triangle. Show that

$$3\cos a\cos b\cos c \leq \cos^2 a+\cos^2 b+\cos^2 c \leq 1+2\cos a\cos b\cos c\,.$$

Solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

By the AM–GM Inequality,

$$\begin{aligned}\cos^2 a+\cos^2 b+\cos^2 c &\geq 3\sqrt[3]{\cos^2 a\cos^2 b\cos^2 c}\\ &\geq 3\sqrt[3]{\cos^3 a\cos^3 b\cos^3 c} = 3\cos a\cos b\cos c\,,\end{aligned}$$

which establishes the left inequality.

[*Ed.:* Next, we give the argument of Benito, Ciaurri, and Fernández for the right inequality.] Let A, B, and C be the vertices of the spherical triangle. We use the well-known fundamental formula of spherical trigonometry,

$$\cos a \;=\; \cos b\cos c + \sin b\sin c\cos A\,.$$

We have

$$\begin{aligned} 0 &\le \sin^2 b\sin^2 c\sin^2 A \;=\; \sin^2 b\sin^2 c - \sin^2 b\sin^2 c\cos^2 A \\ &= (1-\cos^2 b)(1-\cos^2 c) - (\cos a - \cos b\cos c)^2 \\ &= 1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c\,, \end{aligned}$$

which completes the proof.

[*Ed.:* We now proceed with Zhou's argument for the same inequality.] We may assume that the spherical triangle is spanned by the unit vectors $\overrightarrow{A}$, $\overrightarrow{B}$, and $\overrightarrow{C}$, starting at the centre of the sphere. Let $\overrightarrow{A} = \langle A_1, A_2, A_3\rangle$, $\overrightarrow{B} = \langle B_1, B_2, B_3\rangle$, $\overrightarrow{C} = \langle C_1, C_2, C_3\rangle$, and $M = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$. Then

$$\begin{aligned} 0 &\le [(\overrightarrow{A}\times\overrightarrow{B})\cdot\overrightarrow{C}]^2 \;=\; (\det M)^2 \;=\; \det(MM^T) \\ &= \begin{vmatrix} \overrightarrow{A}\cdot\overrightarrow{A} & \overrightarrow{A}\cdot\overrightarrow{B} & \overrightarrow{A}\cdot\overrightarrow{C} \\ \overrightarrow{B}\cdot\overrightarrow{A} & \overrightarrow{B}\cdot\overrightarrow{B} & \overrightarrow{B}\cdot\overrightarrow{C} \\ \overrightarrow{C}\cdot\overrightarrow{A} & \overrightarrow{C}\cdot\overrightarrow{B} & \overrightarrow{C}\cdot\overrightarrow{C} \end{vmatrix} \;=\; \begin{vmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{vmatrix} \\ &= 1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c\,, \end{aligned}$$

completing the proof.

Also solved by the proposer.

KLAMKIN–13. [2005 : 329, 332] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $\mathcal{C}$ be a smooth closed convex curve in the plane. Theorems in analysis assure us that there is at least one circumscribing triangle $A_0B_0C_0$ to $\mathcal{C}$ having minimum perimeter. Prove that the excircles of $A_0B_0C_0$ are tangent to $\mathcal{C}$.

Similar solutions by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

Suppose, to the contrary, that $A_0B_0C_0$ is a circumscribed triangle with the minimum perimeter, but the excircle opposite A_0 does not touch $\mathcal{C}$. Since $\mathcal{C}$ is convex, the common tangent B_0C_0 separates it from the excircle. By means of a dilatation with centre A, we can therefore shrink that circle to a smaller circle Γ that is tangent to $\mathcal{C}$ and to the lines A_0B_0 and A_0C_0. (Note that $\mathcal{C}$ is inside the finite region bounded by Γ and the two lines.) Let the common tangent to $\mathcal{C}$ and Γ at the point where they touch (which is well defined, since $\mathcal{C}$ is convex while Γ is a circle) meet lines A_0B_0 and A_0C_0 at B_1 and C_1. Triangle $A_0B_1C_1$ is then a triangle circumscribed about $\mathcal{C}$, but we have decreased the length of the tangent from A_0 to the opposite excircle. Since that length is equal to the semiperimeter of the triangle (see, for example, Coxeter and Greitzer, *Geometry Revisited*, page 13), we have thereby decreased the perimeter of the triangle. This contradicts the initial assumption that our original triangle had the minimum perimeter. Thus, the excircles of $A_0B_0C_0$ are tangent to $\mathcal{C}$.

Also solved by LI ZHOU, Polk Community College, Winter Haven, FL, USA; the proposer provided a second solution that made use of differential geometry.

KLAMKIN–14. [2005 : 329, 332] *Proposed by Andy Liu, University of Alberta, Edmonton, AB.*

A vertical wall OY meets the horizontal floor OX at the corner O. Initially, a ladder AB is placed so that its bottom B is at O while its apex A is on the wall OY. A cat jumps onto the ladder and clings to the point C where $BC = \lambda AC$ for some real number $0 < \lambda < 1$. This jiggles the ladder so that it begins to slide, with A moving down towards O along YO and B moving away from O along OX, until it comes to rest with A at O and B on OX. What is the curve traced out by the cat?

Solution by Joel Schlosberg, Bayside, NY, USA.

Let C_x and C_y be the projections of C onto OX and OY, respectively. In a Cartesian coordinate system with OX and OY as the x– and y–axes, respectively, let the coordinates of C be (x, y). Let $\angle ABO = \theta$; since $CC_y \parallel BO$, we see that $\angle ACC_y = \angle CBC_x = \theta$. Then $x = AC\cos\theta$, $y = BC\sin\theta$, and

$$\frac{x^2}{AC^2} + \frac{y^2}{BC^2} = \cos^2\theta + \sin^2\theta = 1,$$

or

$$x^2 + \frac{y^2}{\lambda^2} = AC^2.$$

Therefore, the curve traced out by the cat is the portion of an ellipse in the first quadrant (x, $y > 0$) with centre O, major axis along OX (since $\lambda < 1$), and eccentricity $\sqrt{1-\lambda^2}$.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

KLAMKIN–15. [2005 : 329, 332] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

A square $ABCD$ sits in the plane with corners A, B, C, D initially located at positions $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, respectively. The square is rotated counterclockwise through an angle θ ($0^\circ \le \theta < 360^\circ$) four times, with the centre of rotation at the points A, B, C, D in successive rotations. Suppose point A ends up on the x-axis or y-axis. Find all possible values of θ.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

For $j = 0, 1, 2, 3, 4$, let $z_{j,1}$, $z_{j,2}$, $z_{j,3}$, $z_{j,4}$ be the complex numbers representing the corners A, B, C, D, respectively, after the j^{th} rotation. Then $z_{0,1} = 0$, $z_{0,2} = 1$, $z_{0,3} = 1+i$, $z_{0,4} = i$, and for $1 \le j,\, k \le 4$, we have

$$z_{j,k} \;=\; z_{j-1,j} + (z_{j-1,k} - z_{j-1,j})e^{\theta i}\,.$$

Hence,

$$\begin{aligned}
z_{4,1} &= z_{3,4} + (z_{3,1} - z_{3,4})e^{\theta i} \;=\; z_{2,3} + (z_{2,4} - z_{2,3})e^{\theta i} + (z_{2,1} - z_{2,4})e^{2\theta i} \\
&= z_{1,2} + (z_{1,3} - z_{1,2})e^{\theta i} + (z_{1,4} - z_{1,3})e^{2\theta i} + (z_{1,1} - z_{1,4})e^{3\theta i} \\
&= z_{0,1} + (z_{0,2} - z_{0,1})e^{\theta i} + (z_{0,3} - z_{0,2})e^{2\theta i} \\
&\qquad\qquad + (z_{0,4} - z_{0,3})e^{3\theta i} + (z_{0,1} - z_{0,4})e^{4\theta i} \\
&= e^{\theta i} + ie^{2\theta i} - e^{3\theta i} - ie^{4\theta i} \\
&= \cos\theta - \sin 2\theta - \cos 3\theta + \sin 4\theta \\
&\qquad\qquad + i(\sin\theta + \cos 2\theta - \sin 3\theta - \cos 4\theta)\,.
\end{aligned}$$

By the sum-to-product identities,

$$\begin{aligned}
\cos\theta - \cos 3\theta - \sin 2\theta + \sin 4\theta &= 2\sin 2\theta\sin\theta + 2\sin\theta\cos 3\theta \\
&= 2\sin\theta\left[\sin 2\theta + \sin\left(\tfrac{\pi}{2} - 3\theta\right)\right] \\
&= 4\sin\theta\sin\left(\tfrac{\pi}{4} - \tfrac{1}{2}\theta\right)\cos\left(-\tfrac{\pi}{4} + \tfrac{5}{2}\theta\right),
\end{aligned}$$

which equals 0 when $\theta \in \left\{0,\, \pi,\, \frac{\pi}{2},\, \frac{3\pi}{10},\, \frac{7\pi}{10},\, \frac{11\pi}{10},\, \frac{3\pi}{2},\, \frac{19\pi}{10}\right\}$. Similarly,

$$\begin{aligned}
\sin\theta - \sin 3\theta + \cos 2\theta - \cos 4\theta &= -2\sin\theta\cos 2\theta + 2\sin 3\theta\sin\theta \\
&= 2\sin\theta\left[\sin 3\theta - \sin\left(\tfrac{\pi}{2} - 2\theta\right)\right] \\
&= 4\sin\theta\sin\left(-\tfrac{\pi}{4} + \tfrac{5}{2}\theta\right)\cos\left(\tfrac{\pi}{4} + \tfrac{1}{2}\theta\right),
\end{aligned}$$

which equals 0 when $\theta \in \left\{0,\, \pi,\, \frac{\pi}{10},\, \frac{\pi}{2},\, \frac{9\pi}{10},\, \frac{13\pi}{10},\, \frac{17\pi}{10}\right\}$.

In conclusion, the set of values of θ is

$$\{0,\, \pi\} \cup \left\{ \frac{(2n+1)\pi}{10} \;\middle|\; n = 0, 1, 2, \ldots, 9 \right\}.$$

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer. There was one incorrect submission.