Non-Transitivity in Tournaments

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From August 1 to 9, 2005, we attended the International Mathematics Tournament of the Towns Summer Seminar, held in Mir Town, Belarus. Sixty students worked on six research projects. One of them, "The Mathematics of Tournaments", was presented by A. Zaslavsky and B. Frenkin, with contributions from A. Tolpygo and G. Tokarev. Our main results were the proofs of Theorems 2 and 4 below, as well as the determination of the parameters α and β used in the definition of the non-transitivity index. We later learned that our work generalizes that of others, such as [1].

A round-robin tournament, or tournament for short, is a competition in which each pair of players plays each other exactly once. For the moment, we do not allow draws. Thus, each game is a win for one of the players. We can record the results in a complete directed graph as follows. Let the participants be represented by vertices. An arrow is drawn from vertex X to vertex Y if player X beats player Y. This graph is also called a tournament.

Three vertices X, Y, and Z are said to form a transitive triple if, whenever X beats Y and Y beats Z, then X must beat Z. A tournament is said to be transitive if every triple is transitive.

Suppose X_i beats X_j in a transitive tournament with n players. By transitivity, X_i beats every player beaten by X_j. Thus, the players may be labelled X_1, X_2, $\dots$, X_n so that X_i beats X_j if and only if $i < j$. Hence, there is essentially only one transitive tournament for each fixed value of n.

In a non-transitive tournament, there must be some triple of vertices which is non-transitive. Such vertices form a directed 3–cycle. Hence, the number of such 3–cycles is a measure of how non-transitive a tournament is. This number is defined as the non-transitivity index λ of the tournament.

Theorem 1 If a tournament has an m–cycle, then $\lambda \geq m-2$.

Proof: We use induction on m. This is trivial for $m = 3$. Suppose the result holds for some $m \geq 3$. Consider a tournament with an $(m+1)$–cycle $(1, 2, \dots, m+1)$. For $1 \leq i \leq m+1$, consider the arrow between i and $i-2$ (identifying -1 with m and 0 with $m+1$). If, for each $i = 1, \dots, m+1$, the arrow between i and $i-2$ goes from i to $i-2$, then the number of 3–cycles $(i-2, i-1, i)$ is $m+1$, since we have arrows going from $i-2$ to $i-1$ and from $i-1$ to i along the $(m+1)$–cycle. In this case we are done. Suppose instead that, for some i, the arrow between i and $i-2$ goes from $i-2$ to i. By cyclic symmetry, we may assume that this arrow goes from m to 1. Then we have an m–cycle $(1, 2, \dots, m)$. By the induction hypothesis, the number of 3–cycles without the vertex $m+1$ is at least $m-2$. Let k be the highest value such that there is an arrow going from $m+1$ to k. Then $1 \leq k$ since we have an arrow going from $m+1$ to 1, and $k \leq m-1$ since the arrow between m and $m+1$ goes from m to $m+1$. Now $(k, k+1, m+1)$ is another 3–cycle, bringing the total number of 3–cycles to at least $m-1$. ■

An *elbow* in a directed graph is defined as the union of two arrows with a common vertex, which is called the *pivot* of the elbow. There are three kinds of elbows. If both arrows go out from the pivot, it is called an *arrow-from-arrow*; if both arrows go into the pivot, it is called an *arrow-to-arrow*; otherwise, it is called a *broken arrow*. The elbows, especially the broken arrows, will play a central role in our study.

Theorem 2 In a tournament with n players, let the number of wins of the i^{th} player be w_i. Then

$$\lambda = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum_{i=1}^{n} w_i^2 .$$

Proof: The vertex X_i has w_i outgoing arrows and $n-1-w_i$ incoming arrows, and is the pivot of exactly $w_i(n-1-w_i)$ broken arrows. Hence, the total number of broken arrows is

$$\sum_{i-1}^{n} w_i(n-1-w_i) = (n-1)\binom{n}{2} - \sum_{i=1}^{n} w_i^2 .$$

The number of 3–cycles containing 3 broken arrows is λ, while each transitive triple contains only 1. It follows that the total number of broken arrows is also given by $3\lambda + \binom{n}{3} - \lambda$. Equating the two expression above and simplifying, we have the desired result. ■

Theorem 3 In a tournament with n players, we have $\lambda \leq \frac{n^3-n}{24}$ if n is odd and $\lambda \leq \frac{n^3-4n}{24}$ if n is even.

Proof: By Theorem 2, $\lambda = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum_{i=1}^{n} w_i^2$. This remains unchanged if the result of every game is reversed. Therefore, we also have $\lambda = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum_{i=1}^{n}(n-1-w_i)^2$. Together, these yield

$$\begin{aligned}
\lambda &= \frac{n(n-1)(2n-1)}{12} - \frac{1}{4}\sum_{i=1}^{n}\left(w_i^2 + (n-1-w_i)^2\right) \\
&= \frac{n(n-1)(2n-1)}{12} - \frac{1}{8}\sum_{i=1}^{n}\left((n-1)^2 + (2w_i - n + 1)^2\right) \\
&= \frac{n^3-n}{24} - \frac{1}{8}\sum_{i=1}^{n}(2w_i - n + 1)^2 .
\end{aligned}$$

If n is odd, the maximum occurs when $w_i = (n-1)/2$ for all i, yielding $\lambda \leq (n^3-n)/24$. If n is even, the maximum occurs when $w_i = n/2$ for $n/2$ players and $w_i = n/2 - 1$ for $n/2$ players, yielding $\lambda \leq (n^3-4n)/24$. ■

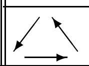
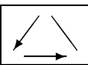
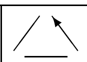
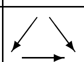
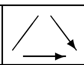
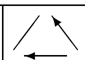
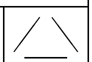
We now allow draws in our tournaments. If the game between players X and Y is a draw, we join vertices X and Y by an ordinary edge as opposed to an arrow. The resulting graph, still called a tournament, is neither an ordinary graph nor a directed graph, but a mixed graph with both edges and arrows.

There are now three additional kinds of transitive triples. In one, X draws with Y, Y draws with Z, and X draws with Z. In the other two kinds, X draws with Y while both beat Z or both are beaten by Z. These triples are vacuously transitive since the question of transitivity never arises. As before, a tournament is said to be transitive if every triple is transitive. However, there are now many transitive tournaments with the same number of players. In addition to the unique transitive tournament without draws, there is at least the transitive tournament in which all games are draws.

The non-transitivity index λ is no longer equal to the number of 3–cycles, because we have two additional kinds of non-transitive triples. In one, X beats Y, Y beats Z, but X draws with Z. In the other kind, X draws with Y, Y draws with Z, but X beats Z. Let their numbers be y and z respectively, and let x be the number of 3–cycles. It would appear that $\lambda = x + y + z$ is a natural definition. However, the degree of non-transitivity represented by each kind of non-transitive triple is not quite the same. Thus, we define $\lambda = x + \alpha y + \beta z$ for some suitably chosen parameters α and β.

How should α and β be chosen? Intuitively, one would think that the value of α should be $\frac{1}{2}$ since replacing an arrow in the wrong direction by an edge makes the non-transitivity only half as bad. As it turned out, our intuition does not lead us astray. However, it would not be so easy to divine a suitable value for β.

We turn once again to the elbows, of which there are three additional kinds, because either of the arrows may be replaced by an ordinary edge. An *arrow-from-edge* consists of an edge and an arrow going out from the pivot. An *arrow-to-edge* consists of an edge and an arrow going into the pivot. A *broken edge* consists of two edges hinged by a pivot. As in the proof of Theorem 2, we count the number of each kind of elbows in each kind of triples. The result is shown in the chart below.

Elbows	Non-transitive Triples			Transitive Triples			
							
Broken Arrow	3	1	0	1	0	0	0
Arrow-from-Arrow	0	0	0	1	0	1	0
Arrow-to-Arrow	0	0	0	1	1	0	0
Arrow-from-Edge	0	1	1	0	2	0	0
Arrow-to-Edge	0	1	1	0	0	2	0
Broken Edge	0	0	1	0	0	0	3

Let a, b, c, and d be the respective numbers of the four kinds of transitive triples. The total numbers of the six kinds of elbows are given by $3x+y+a$, $a+c$, $a+b$, $y+z+2b$, $y+z+2c$, and $z+3d$, respectively. We seek an expression involving only x, y, and z. Multiplying these six expressions by 4, -2, -2, 1, 1, and 0, respectively, and then adding, we have $12x+6y+2z$. Rewriting this as $12(x+\frac{1}{2}y+\frac{1}{6}z)$, we see that sensible choices are $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{6}$.

Theorem 4 In a tournament with n players, let the numbers of wins, draws, and losses of the i^{th} player be w_i, d_i, and ℓ_i, respectively. Then

$$\lambda = \frac{n(n-1)(n+1)}{24} - \frac{1}{24}\sum_{i=1}^{n} d_i(d_i+2) - \frac{1}{8}\sum_{i=1}^{n}(w_i-\ell_i)^2 .$$

Proof: From the chart,

$$3x+y+a = \sum_{i-1}^{n} w_i\ell_i , \qquad a+c = \sum_{i=1}^{n}\binom{w_i}{2},$$

$$a+b = \sum_{i=1}^{n}\binom{\ell_i}{2}, \qquad y+z+2b = \sum_{i=1}^{n} w_i d_i ,$$

$$y+z+2c = \sum_{i=1}^{n} d_i\ell_i , \qquad z+3d = \sum_{i=1}^{n}\binom{d_i}{2}.$$

Multiplying these by 8, -4, -4, 2, 2, and 0, respectively, we have

$$\begin{aligned}
24\lambda &= 24x+12y+4z \\
&= 8\sum_{i=1}^{n} w_i\ell_i - 4\sum_{i=1}^{n}\left(\binom{w_i}{2}+\binom{\ell_i}{2}\right) + 2\sum_{i=1}^{n} d_i(w_i+\ell_i) \\
&= \sum_{i=1}^{n}(w_i+d_i+\ell_i)^2 + 2\sum_{i=1}^{n}(w_i+d_i+\ell_i) - \sum_{i=1}^{n} d_i^2 - 2\sum_{i=1}^{n} d_i \\
&\qquad + 6\sum_{i=1}^{n} w_i\ell_i - 3\sum_{i=1}^{n} w_i^2 - 3\sum_{i=1}^{n}\ell_i^2 \\
&= n(n-1)^2 + 2n(n-1) - \sum_{i=1}^{2} d_i(d_i+2) - 3\sum_{i=1}^{n}(w_i-\ell_i)^2 .
\end{aligned}$$

The desired result follows immediately. ∎

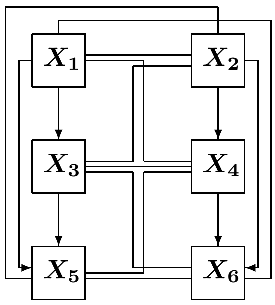
It follows from Theorem 4 that the result in Theorem 3 also holds for tournaments with draws.

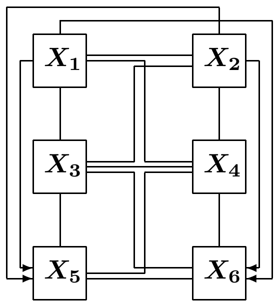
A tournament is said to be *semi-transitive* if whenever X beats Y and Y beats Z, then X must beat Z. In other words, the third kind of non-transitive triple is acceptable. In a tournament with draws, the score s of a player with w wins, d draws, and ℓ losses is given by $s=w+\dfrac{d}{2}$.

Theorem 5 For any tournament without draws, there exists a semi-transitive tournament with draws involving the same players such that each has the same score in both tournaments.

Proof: Among all tournaments involving the same players in which each has the same score, consider the one with the highest number of draws. We claim that it is semi-transitive. If not, then there would exist a triple where X beats Y, Y beats Z, and either Z beats X or Z draws with X. In the former case, we replace all three games with draws. In the latter case, let X draw with Y, Y draw with Z and X beat Z instead. These changes do not affect the individual scores but increase the number of draws, contradicting our maximality assumption. ■

We close this paper with two questions. Consider two semi-transitive tournaments involving the same players in which each has the same score. Is it necessarily true that both tournaments have (i) the same number of draws, or (ii) the same number of non-transitive triples?





Answers. In each of the two tournaments above, X_1 and X_2 have scores of $2\frac{1}{2}$, X_3 and X_4 have scores of 2 while X_5 and X_6 have scores of $1\frac{1}{2}$. Yet there are 9 draws and 18 non-transitive triples in the first but 11 draws and 8 non-transitive triples in the second. Thus, the answer to both questions is negative.

Reference

[1] M.G. Kendall and B. Babbington Smith, On the Method of Paired Comparisons, *Biometrika* **31** (1940), 324–345.

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