THE OLYMPIAD CORNER

No. 255

R.E. Woodrow

Welcome back from the Summer break (at least in Canada!). To start your problem-solving engines we give the 2003 Belarus Mathematical Olympiad. Thanks go to Andy Liu, Canadian Team Leader to the 2003 IMO in Japan, for supplying us with these problems.

BELARUS MATHEMATICAL OLYMPIAD 2003
June 21-22, 2003

1. Let T be the set of all ordered triples of non-negative integers. Find all functions f from T to the real numbers such that $f(x,y,z) = 0$ when $xyz = 0$, and, when $xyz \neq 0$,

$$\begin{aligned} f(x,y,z) \\ = \; & 1 + \tfrac{1}{6}\big[f(x+1,y-1,z) + f(x-1,y+1,z) + f(x+1,y,z-1) \\ & + f(x-1,y,z+1) + f(x,y+1,z-1) + f(x,y-1,z+1)\big]\,. \end{aligned}$$

2. Define a *k–clique* to be a set of k people each of whom is acquainted with all of the others. At a certain party, every pair of 3–cliques has at least one person in common, and there are no 5–cliques. Prove that there are two or fewer people at the party whose departure leaves no 3–cliques.

3. Find all functions f from the real numbers to the real numbers such that, for any real numbers x and y,

$$f(xy)\big(f(x) - f(y)\big) \;=\; (x-y)f(x)f(y)\,.$$

4. From a point P outside triangle ABC, the feet of the perpendiculars to BC, CA, and AB are D, E, and F, respectively. If triangles PAF, PBD, and PCE all have equal area, prove that ABC also has the same area.

5. Twenty-one girls and twenty-one boys took part in a mathematics competition. Each contestant solved at most six problems. For any pair consisting of a girl and a boy, there was at least one problem solved by both contestants. Prove that there was a problem solved by at least three girls and at least three boys.

6. The sequence $\{a_n\}$ is defined by $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$ and $a_{n+3} = |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n|$ for all non-negative integers n. Determine a_n when $n = 14^{14}$.

As a second set of problems we give the Selection Problems for the Indian Team for IMO 2003. Thanks again go to Andy Liu, Canadian Team Leader, for collecting them for our use.

PROBLEMS TO SELECT INDIAN IMO TEAM 2003

1. Let A', B', C' be the mid-points of the sides BC, CA, AB, respectively, of an acute non-isoceles triangle ABC, and let D, E, F be the feet of the altitudes through the vertices A, B, C on these sides, respectively. Consider the arc DA' of the nine-point circle of triangle ABC lying outside the triangle. Let the point of trisection of this arc closer to A' be A''. Define analogously the points B'' (on arc EB') and C'' (on arc FC'). Show that triangle $A''B''C''$ is equilateral.

2. Find all triples (a, b, c) of positive integers such that

(i) $a \le b \le c$;

(ii) $\gcd(a, b, c) = 1$; and

(iii) $a^3 + b^3 + c^3$ is divisible by each of the numbers a^2b, b^2c, c^2a.

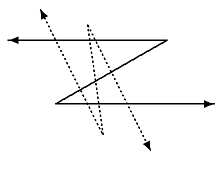
3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all x, y in $\mathbb{R}$, we have

$$f(x+y) + f(x)f(y) \;=\; f(x) + f(y) + f(xy) \,.$$

4. There are four lines in the plane, no three concurrent, no two parallel, and no three forming an equilateral triangle. If one of them is parallel to the *Euler line* of the triangle formed by the other three lines, prove that a similar statement holds for each of the other lines.

5. On the real number line, paint red all points that correspond to integers of the form $81x + 100y$, where x and y are positive integers. Paint the remaining integer points blue. Find a point P on the line such that, for every integer point T, the reflection of T with respect to P is an integer point of a different colour than T.

6. A zig-zag in the plane consists of two parallel half-lines connected by a line segment. Find z_n, the maximum number of regions into which n zig-zags can divide the plane. For example, $z_1 = 2$ and $z_2 = 12$ (see the diagram). Of these z_n regions, how many are bounded? [The zig-zags can be as narrow as you please.] Express your answers as polynomials in n of degree not exceeding 2.



7. Let $P(x)$ be a polynomial with integer coefficients such that $P(n) > n$ for all positive integers n. Suppose that for each positive integer m, there is a term in the sequence $P(1)$, $P(P(1))$, $P(P(P(1)))$, $\ldots$ which is divisible by m. Show that $P(x) = x + 1$.

8. Let ABC be a triangle, and let r, r_1 ,r_2, r_3 denote its inradius and the exradii opposite the vertices A, B, C, respectively. Suppose $a > r_1$, $b > r_2$, $c > r_3$. Prove that

(a) triangle ABC is acute, (b) $a + b + c > r + r_1 + r_2 + r_3$.

9. Let n be a positive integer and $\{A, B, C\}$ a partition of $\{1, 2, \ldots, 3n\}$ such that $|A| = |B| = |C| = n$. Prove that there exist $x \in A$, $y \in B$, $z \in C$ such that one of x, y, z is the sum of the other two.

10. Let n be a positive integer greater than 1, and let p be a prime such that n divides $p - 1$ and p divides $n^3 - 1$. Prove that $4p - 3$ is a square.

As a final set of problems for this issue we give the 2003 German Mathematical Olympiad, Final Round, Grades 12-13.

GERMAN MATHEMATICAL OLYMPIAD 2003

Final Round, Grades 12-13

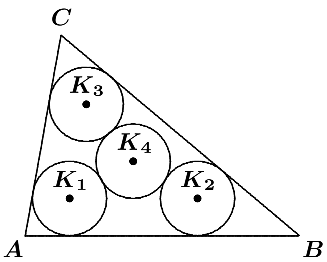
June 22-25, 2003

1. Determine all pairs (x, y) of real numbers x, y which satisfy

$$\begin{aligned} x^3 + y^3 &= 7\,, \\ xy(x + y) &= -2\,. \end{aligned}$$

2. In the interior of a triangle ABC, circles K_1, K_2, K_3, and K_4 of the same radii are defined such that K_1, K_2, and K_3 touch two sides of the triangle and K_4 touches K_1, K_2, and K_3, as shown in the figure.

Prove that the centre of K_4 is located on the line through the incentre and the circumcentre.

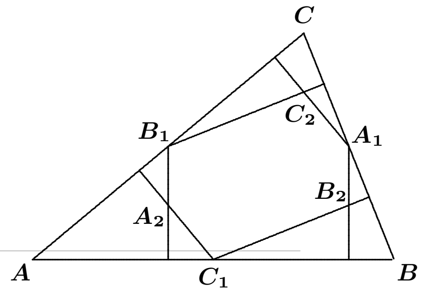


3. The caterpillar *Nummersatt* is sitting in the middle square of an $N \times N$ board, where N is an odd integer with $N \geq 3$. The other squares of the board each contain a positive integer, and all of these integers are different. Nummersatt wants to find a way off the board. The caterpillar can move only between adjacent squares (squares having a common side), or off the board from one of the outermost squares, having once reached such a square. On reaching a new square, Nummersatt has to eat the number on that square. The number n weighs $\frac{1}{n}$ kg, and Nummersatt cannot eat more than 2 kg.

Decide whether numbers can be distributed on the board so that there is no way off the board for Nummersatt

(a) for $N = 2003$, (b) for all odd integers $N \geq 3$.

4. Let A_1, B_1, and C_1 be the mid-points of the sides of the acute-angled triangle ABC. The 6 lines through these points perpendicular to the other sides meet in the points A_2, B_2, and C_2, as shown in the figure. Prove that the area of the hexagon $A_1C_2B_1A_2C_1B_2$ equals half of the area of $\triangle ABC$.



5. If n is a positive integer, let $a(n)$ be the smallest positive number for which $(a(n))!$ is divisible by n. Determine all positive integers n satisfying

$$\frac{a(n)}{n} = \frac{2}{3}.$$

6. Prove that there are infinitely many pairs (a, b) of positive integers with $a > b$ having the following properties:

(i) the greatest common divisor of a and b equals 1;

(ii) a is a divisor of $b^2 - 5$.

(iii) b is a divisor of $a^2 - 5$.

Now we turn to solutions from our readers to some of the problems from the February 2005 *Corner*. The first group of solutions are to problems of the Icelandic Mathematical Contest 2000-2001 given in [2005 : 26–27].

1. Let x and y be positive real numbers such that $xy = 1$. Prove that

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

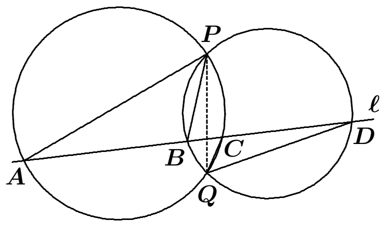
Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; R. Laumen, Deurne, Belgium; Geoffrey A. Kandall, Hamden, CT, USA; and Vedula N. Murty, Dover, PA, USA. We give Kandall's write-up.

The condition $xy = 1$ is unnecessary. For any positive real numbers x and y,

$$\frac{x}{y} + \frac{y}{x} = \left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^2 + 2 \geq 2.$$

2. Two circles intersect at points P and Q. A line ℓ that intersects the line segment PQ intersects the two circles at the points A, B, C, and D (in that order along the line ℓ). Prove that $\angle APB = \angle CQD$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pierre Bornsztein, Maisons-Laffitte, France; Paolo Custodi, Fara Novarese, Italy; R. Laumen, Deurne, Belgium; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Laumen's write-up.



We have $\angle PBD = \angle PQD$ (since these angles are subtended by the same arc) and $\angle PQD = \angle CQD + \angle CQP$. Hence, $\angle PBD = \angle CQD + \angle CQP$. Also, since $\angle PBD$ is an exterior angle to $\triangle APB$, we have $\angle PBD = \angle APB + \angle PAB$. Thus,

$$\angle APB + \angle PAB = \angle CQD + \angle CQP\,.$$

But $\angle PAB = \angle PAC = \angle CQP$ (since these angles are subtended by the same arc). Therefore, $\angle APB = \angle CQD$.

3. Richard is walking up a stair that has 10 steps. With each stride he goes up either one step or two steps. In how many different ways can Richard go up the stairs?

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Paolo Custodi, Fara Novarese, Italy; and Geoffrey A. Kandall, Hamden, CT, USA. We give Custodi's write-up, expanded by the editor.

To count the number of ways in which Richard climbs 2–step blocks k times, we are really looking at the number of ways of arranging a sequence of k 2–step blocks and $10 - 2k$ 1–step blocks, which is $\binom{10-k}{k}$. Thus, the number of different ways Richard can go up the stairs is

$$\binom{10}{0} + \binom{9}{1} + \binom{8}{2} + \binom{7}{3} + \binom{6}{4} + \binom{5}{5} = 89\,.$$

Generalizing, if there are $2n$ steps, the number of ways is $\sum\limits_{k=0}^{n} \binom{2n-k}{k}$.

4. In Flora's number-set there are the numbers

$$2^n - 1\,,\quad 3^{2n} - 1\,,\quad 4^{3n} - 1\,,\quad 5^{4n} - 1\,,\quad 6^{5n} - 1\,,\quad 7^{6n} - 1\,,\quad 8^{7n} - 1\,,\quad 9^{8n} - 1\,,$$

for each natural number n, and there are no other numbers in the set. How many square numbers does the set contain?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Paolo Custodi, Fara Novarese, Italy; Geoffrey A. Kandall, Hamden, CT, USA; and R. Laumen, Deurne, Belgium. We give Kandall's write-up.

The numbers $3^{2n} - 1$, $5^{4n} - 1$, $7^{6n} - 1$, and $9^{8n} - 1$ can be eliminated, since tbey are 1 less than a square. The numbers $2^n - 1$ for $n \geq 2$, $4^{3n} - 1$,

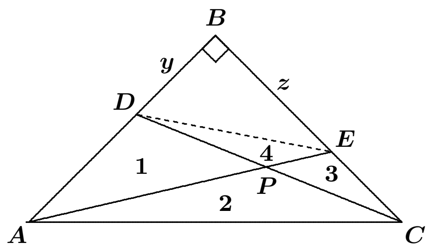
$6^{5n} - 1$, and $8^{7n} - 1$ can be eliminated, since each is congruent to $-1 \pmod 4$, whereas a square must be congruent to 0 or 1 $\pmod 4$.

This leaves only the one square $2^1 - 1 = 1$.

5. Triangle ABC is isosceles with a right angle at B and $AB = BC = x$. Point D on the side AB and point E on the side BC are chosen such that $BD = BE = y$. The line segments AE and CD intersect at the point P. What is the area of the triangle APC, expressed in terms of x and y?

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Paolo Custodi, Fara Novarese, Italy; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON. We give the generalization by Wang and Zhao.

We consider the more general problem of determining the area of $\triangle APC$ if $BD = y$ and $BE = z$. The given problem is the special case where $y = z$.



We connect DE and let $\triangle_1$, $\triangle_2$, $\triangle_3$, $\triangle_4$ denote the areas of the triangles labelled by 1, 2, 3, 4, respectively. We are to determine $\triangle_2$ in terms of x, y, and z.

From the diagram, it is easily seen that

$$\triangle_1 + \triangle_2 = \tfrac{1}{2}x(x-y)\,, \qquad (1)$$

$$\triangle_1 + \triangle_4 = \tfrac{1}{2}z(x-y)\,, \qquad (2)$$

$$\text{and} \quad \triangle_2 + \triangle_3 = \tfrac{1}{2}x(x-z)\,. \qquad (3)$$

Since $\dfrac{\triangle_1}{\triangle_2} = \dfrac{DP}{CP} = \dfrac{\triangle_4}{\triangle_3}$, we have, using (2) and (3),

$$\frac{\triangle_1}{\triangle_2} = \frac{\triangle_1 + \triangle_4}{\triangle_2 + \triangle_3} = \frac{\frac{1}{2}z(x-y)}{\frac{1}{2}x(x-z)}\,;$$

that is, $\triangle_1 = \dfrac{z(x-y)}{x(x-z)}\triangle_2$. Substituting into (1), we get

$$\frac{z(x-y)}{x(x-z)}\triangle_2 + \triangle_2 = \tfrac{1}{2}x(x-y)\,,$$

$$\text{or} \quad \big(z(x-y) + x(x-z)\big)\triangle_2 = \tfrac{1}{2}x^2(x-y)(x-z)\,.$$

Hence, $\triangle_2 = \dfrac{x^2(x-y)(x-z)}{2(x^2-yz)}$.

For the original problem, where $y = z$, we have $\triangle_2 = \dfrac{x^2}{2}\left(\dfrac{x-y}{x+y}\right)$.

Remark. Even more generally, we can remove the assumption that $\angle B$ is a right angle and show, using arguments similar to those given above, that $\triangle_2 = \frac{(x-y)(x-z)}{x^2-yz}\triangle$, where $\triangle$ denotes the area of triangle ABC.

6. How many natural numbers are divisible by 2001 and have exactly 2001 natural divisors?

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Paolo Custodi, Fara Novarese, Italy; Geoffrey A. Kandall, Hamden, CT, USA; R. Laumen, Deurne, Belgium; Vedula N. Murty, Dover, PA, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Custodi's write-up.

Let n be a natural number which is divisible by 2001 and also has 2001 divisors. Decompose n into prime factors as $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, where p_1, p_2, $\dots$, p_k are distinct primes and a_1, a_2, $\dots$, a_k are positive integers. Then the number of divisors of n is

$$\prod_{i=1}^{k}(a_i+1) \;=\; 2001 \;=\; 3\cdot 23\cdot 29\,.$$

Since n is divisible by 2001, each prime factor of 2001 is a factor of n. It follows that $k=3$ and $\{a_1, a_2, a_3\} = \{2, 22, 28\}$. Therefore, we must have $\{p_1, p_2, p_3\} = \{3, 23, 29\}$. Now there are six possibilities for n:

$$3^2\cdot 23^{22}\cdot 29^{28}\,,\qquad 3^2\cdot 23^{28}\cdot 29^{22}\,,\qquad 3^{22}\cdot 23^2\cdot 29^{28}\,,$$
$$3^{22}\cdot 23^{28}\cdot 29^{2}\,,\qquad 3^{28}\cdot 23^{2}\cdot 29^{22}\,,\qquad 3^{28}\cdot 23^{22}\cdot 29^{2}\,.$$

Now we have readers' solutions to problems of the Greek Mathematical Competitions Selection Examination for the IMO 2002 given in [2005 : 27].

2. Let x, y, a be real numbers such that

$$x+y \;=\; x^3+y^3 \;=\; x^5+y^5 \;=\; a\,.$$

Determine all the possible values of a.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's solution.

The possible values of a are 0, 1, -1, 2, and -2.

Observe that $a=0$ when $y=-x$, $a=1$ when $x=0$ and $y=1$, $a=-1$ when $x=0$ and $y=-1$, $a=2$ when $x=y=1$, and $a=-2$ when $x=y=-1$.

Conversely, suppose that $x+y = x^3+y^3 = x^5+y^5 = a \neq 0$ for some real numbers x and y. We show that a must be 1, -1, 2, or -2.

Let $p = xy$. Then $a = x^3 + y^3 = (x+y)^3 - 3xy(x+y) = a^3 - 3ap$, which yields $a^2 = 1 + 3p$. Also,

$$a \;=\; x^5 + y^5 \;=\; (x+y)^5 - 5xy(x^3+y^3) - 10xy(x+y) \;=\; a^5 - 15ap\,,$$

and hence, $a^4 = 1 + 15p$. Thus, $1 + 15p = (1+3p)^2$, which is easily solved to get $p = 0$ or $p = 1$.

If $p = 0$, then the equation $a^2 = 1 + 3p$ gives $a = 1$ or $a = -1$. If $p = 1$, then the same equation gives $a = 2$ or $a = -2$.

4. Prove that the following inequality holds for every triple (a, b, c) of non-negative real numbers with $a^2 + b^2 + c^2 = 1$:

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \;\geq\; \frac{3}{4}\left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right)^2 .$$

When does equality hold?

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Nick Skombris and Babis Stergiou, Chalkida, Greece. We give the solution by Skombris and Stergiou, modified by the editor.

First we note that if x_1, x_2, x_3 are any real numbers and w_1, w_2, w_3 are any positive real numbers, then, by the Cauchy-Schwarz Inequality,

$$\left(\sum_{i=1}^{3} x_i\right)^2 \;=\; \left(\sum_{i=1}^{3} \sqrt{w_i}\frac{x_i}{\sqrt{w_i}}\right)^2 \;\leq\; \left(\sum_{i=1}^{3} w_i\right)\left(\sum_{i=1}^{3} \frac{x_i^2}{w_i}\right) . \tag{1}$$

Now we let $x_1 = a\sqrt{a}$, $x_2 = b\sqrt{b}$, $x_3 = c\sqrt{c}$, $w_1 = a^2(b^2+1)$, $w_2 = b^2(c^2+1)$, and $w_3 = c^2(a^2+1)$. Then

$$\begin{aligned}\sum_{i=1}^{3} \frac{x_i^2}{w_i} &= \frac{\left(a\sqrt{a}\right)^2}{a^2(b^2+1)} + \frac{\left(b\sqrt{b}\right)^2}{b^2(c^2+1)} + \frac{\left(c\sqrt{c}\right)^2}{c^2(a^2+1)} \\ &= \frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1}\,, \qquad (2)\end{aligned}$$

and

$$\sum_{i=1}^{3} w_i \;=\; a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2 \;=\; a^2b^2 + b^2c^2 + c^2a^2 + 1\,.$$

Since

$$a^2b^2 + b^2c^2 + c^2a^2 \;\leq\; a^4 + b^4 + c^4 \;=\; (a^2+b^2+c^2)^2 - 2\left(a^2b^2 + b^2c^2 + c^2a^2\right),$$

we have

$$a^2b^2 + b^2c^2 + c^2a^2 \;\leq\; \tfrac{1}{3}\left(a^2+b^2+c^2\right)^2 \;=\; \tfrac{1}{3}\,.$$

Thus, $\sum\limits_{i=1}^{3} w_i \leq \frac{1}{3} + 1 = \frac{4}{3}$. Using this result along with (2) in (1), we get

$$\left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right)^2 \;\leq\; \frac{4}{3}\left(\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1}\right),$$

which immediately yields the desired result.

We have equality if and only if $a = b = c = \frac{1}{3}\sqrt{3}$.

Next we look at solutions to problems of the 16th China Mathematical Olympiad, Selected Problems, given in [2005 : 28].

2. Let $X = \{1, 2, \ldots, 2001\}$. Find the minimum positive integer m such that, for each m–element subset W of X, there exist $u, v \in W$ (u and v may be the same) with $u + v = 2^k$ for some positive integer k.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We prove that $m = 999$.

Let W be a subset of X such that there do not exist $u, v \in W$ with $u + v$ a power of 2. Consider the pairs of the form:

(a) $(47 + k, 2001 - k)$ for $k = 0$ to 976 (these pairs contain all the integers from 47 to 2001, except 1024, and each pair sums to $2048 = 2^{11}$).

(b) $(18 + k, 46 - k)$ for $k = 0$ to 13 (these pairs contain all the integers from 18 to 46, except 32, and each pair sums to $64 = 2^6$).

(c) $(2 + k, 14 - k)$ for $k = 0$ to 5 (these pairs contain all the integers from 2 to 14, except 8, and each pair sums to $16 = 2^4$).

(d) $(15, 17)$ (this pair sums to $32 = 2^5$).

This leaves 1, 8, 16, 32, 1024 as the only the unpaired integers.

Clearly, W does not contain any of the 5 unpaired integers and contains at most one integer from each pair. Thus,

$$|W| \;\leq\; 2001 - 977 - 14 - 6 - 1 - 5 \;=\; 998\,,$$

which leads to $m \leq 999$.

On the other hand, direct checking shows that if

$$W \;=\; \{9, 10, \ldots, 15, 33, 34, \ldots, 46, 1025, 1026, \ldots, 2001\},$$

then W does not contain two elements which add up to a power of 2, and $|W| = 998$. Therefore $m \geq 999$, and we are done.

3. At each vertex of a regular n-sided polygon, there was a magpie. When scared, all the magpies flew away. After a while they all returned, one to each vertex, but not all to their former positions. Find all positive integers n for which there must be 3 magpies such that the triangle formed by the vertices at which they first stood and the triangle formed by the vertices at which they now stand are both acute triangles, both right triangles, or both obtuse triangles.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We prove that the desired values of n are those such that $n \geq 3$ and $n \neq 5$.

Clearly $n \geq 3$. The case $n = 3$ is trivial since there is only one triangle.

Case 1. n is even.

Let us consider any two diametrically opposite vertices, say A and B, of the polygon. Then, for each other vertex C, the triangle ABC is right-angled. Assume that the magpies which initially stood at A and B now stand at A' and B', respectively.

If A' and B' are diametrically opposite vertices, then, for any third bird, which goes from a vertex C to a vertex C', the triangles ABC and $A'B'C'$ are right-angled. Otherwise, let C' be the vertex diametrically opposite from A'. Then $C' \neq B'$, and the bird which is now standing at C' comes from a vertex $C \neq B$. The triangles ABC and $A'B'C'$ are right-angled again.

This proves that all even values of n (except 2) are solutions.

Case 2. n is odd and $n \geq 5$.

We will prove that, if $n \geq 7$, there are more obtuse than acute triangles among the triangles whose vertices are vertices of the polygon (note that there is no right triangle). Assume we have proved this assertion. Then it is impossible to transform each obtuse triangle into an acute one, so that there necessarily exists an obtuse triangle which is transformed into another obtuse triangle. In that case, n is a solution of the problem.

Now let us prove the assertion. Let $n = 2k+1$, and let $A_1A_2\ldots A_{2k+1}$ be the polygon. Then the number of triangles having vertices among $\{A_1, A_2, \ldots, A_{2k+1}\}$ is

$$T_n = \frac{n(n-1)(n-2)}{6} = \frac{2k(2k+1)(2k-1)}{6}.$$

The triangle $A_iA_1A_j$ (with $i < j$) has an obtuse angle at A_1 if and only if $i \in \{2, \ldots, k+1\}$ and $j \in \{k+1+i, \ldots, 2k+1\}$. Thus, for a given $i \in \{2, \ldots, k+1\}$, there are $k+1-i$ admissible values for j. It follows that there are $k(k-1)/2$ obtuse triangles with obtuse angle at A_1. The same reasoning applies to the other vertices. Since each obtuse triangle has only one obtuse angle, the total number of obtuse triangles is

$$O_n = n \cdot k(k-1)/2 = (2k+1)k(k-1)/2.$$

Now, it is straightforward to verify that $O_n > \frac{1}{2}T_n$ for $k \geq 3$, which is $n \geq 7$. This proves the claim.

Case 3. $n = 5$.

In this case, the existence is not assured, as the following example shows: Assume that the magpies are initially at positions $M_1M_2M_3M_4M_5$, and that after being scared they come back at positions $M_1M_4M_2M_5M_3$.

Thus, $n = 5$ is not a solution.

4. Let a, b, c, $a+b-c$, $a+c-b$, $b+c-a$, $a+b+c$ be 7 distinct prime numbers such that the sum of two of a, b, c is 800. Let d be the difference between the largest and smallest numbers among the 7 primes. Find the largest possible value of d.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The largest possible value of d is 1594.

First note that if one of a, b, c is 2, say $a = 2$, then b and c are odd, so that $a+b+c$ is even and greater than 2, which contradicts the hypothesis that it is a prime. Thus $a, b, c \geq 3$ and all the seven primes are odd.

Without loss of generality, we may assume that $a+b = 800$ and $a < b$. Since $a+b-c \geq 3$ (since it is an odd prime number), we deduce that $c \leq 797$. Clearly, the greatest of the seven primes is $a+b+c$. Therefore,

$$d \leq (a+b+c)-3 \leq 800+797-3 = 1594\,.$$

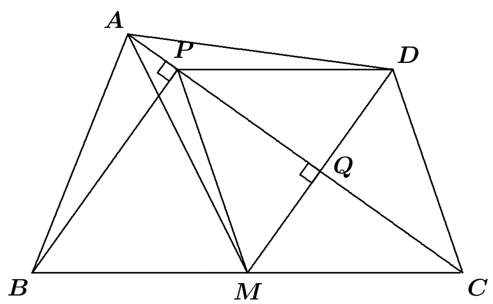
Conversely, note that $800+797 = 1597$ is a prime. And 797 is a prime too. Thus, if $a = 13$, $b = 787$ and $c = 797$ it follows that $a+b+c = 1597$, $a+b-c = 23$, $a+c-b = 3$ and $b+c-a = 1571$ are all primes. And in that case $d = 1594$.

Now we return to the April 2005 number and solutions to problems of the XX Colombian Mathematical Olympiad, Higher Level, which appeared in [2005 : 150–151].

1. [7 points] Let ABC be an isosceles triangle with $AB = AC$. Let M be the mid-point of side BC. The circle with diameter AB cuts side AC at point P. The parallelogram $MPDC$ is constructed so that $PD = MC$ and $PD \parallel MC$. Prove that triangles APD and APM are congruent.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

It is not necessary to assume that $\triangle ABC$ is isosceles.



Let Q be the point of intersection of AC and MD. Since $MPDC$ is a parallelogram, we have $MQ = QD$. Also, $BPDM$ is a parallelogram; in particular, $BP \parallel MD$. Since $\angle APB$ is a right angle, so is $\angle AQM$. Thus, AC is the perpendicular bisector of MD. Therefore, $AM = AD$ and $PM = PD$, and it follows that $\triangle APD \cong \triangle APM$ by SSS.

2. [7 points] Find all positive integers z for which the equation

$$x(x+z) \;=\; y^2$$

has no solutions x, y that are positive integers.

Solved by Houda Anoun, Bordeaux, France; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's solution.

We will say that a positive integer z is *bad* if the equation

$$x(x+z) \;=\; y^2 \qquad (1)$$

has no solution in positive integers (and z is *good* otherwise). We will prove that the bad integers, for which the problem is asking, are 1, 2 and 4.

First note that if (x,y,z) is a solution of (1) in positive integers, then $(2x,2y,2z)$ is also a solution. Therefore, if z is good, then $2z$ is also good.

If $z=2n+1$, with $n \geq 1$, then z is good because it suffices to choose $x=n^2$ and $y=n(n+1)$ to obtain a solution of (1) in positive integers. It follows that each positive integer which is not a power of 2 is good.

For each positive integer x, we have $x^2 < x(x+4) < (x+2)^2$. If $x(x+4)=y^2$, then we must have $x(x+4)=(x+1)^2$. But this equation is equivalent to $2x=1$, which has no integer solutions. Thus, $z=4$ is bad, from which we deduce that 1, 2, and 4 are bad.

Clearly, $(1,3,8)$ is a solution of (1), which means that $z=8$ is good. Then $z=2^n$ is good for all integers $n \geq 3$, and the proof is complete.

3. [7 points] Let $n \geq 4$ be a fixed integer. Let $S=\{P_1,\, P_2,\, \dots,\, P_n\}$ be a set of n points in the plane, no three of which are collinear and no four concyclic. Let a_t, $1 \leq t \leq n$, be the number of circles $P_iP_jP_k$ that contain P_t in the interior, and let

$$m(S) \;=\; a_1+a_2+\cdots+a_n\,.$$

Prove that there exists a positive integer $f(n)$, depending only on n, such that the points of S are the vertices of a convex polygon if and only if $m(S)=f(n)$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that $f(n)=2\binom{n}{4}=\frac{1}{12}n(n-1)(n-2)(n-3)$.

Lemma. If $A_1,\,\dots,\,A_n$ are $n \geq 4$ points of the plane, no three of which are collinear, and such that any four of them are always the vertices of a convex quadrilateral, then they are the vertices (in some order) of a convex n–gon.

Proof of the lemma. Suppose, for the purpose of contradiction, that the convex hull $\mathcal{C}$ of the n points is not an n–gon. Then at least one of the points, say A_1, is an interior point of $\mathcal{C}$. Without loss of generality, we may assume that A_2 is a vertex of $\mathcal{C}$. We triangulate $\mathcal{C}$ by using the diagonals from A_2. Since no three points are collinear, we deduce that A_1 is an interior point of one of the triangles, say $A_2A_3A_4$. But this contradicts the hypothesis that A_1, A_2, A_3, A_4 are the vertices of a convex quadrilateral. ∎

Returning to the problem, let A, B, C, D be any four distinct points from S. Let $n_A = 1$ if the circumcircle of the triangle BCD contains A, and $n_A = 0$ otherwise. The numbers n_B, n_C, n_D are defined similarly. Since no three are collinear, the convex hull of the four points is either a triangle or a quadrilateral.

Case 1. The convex hull is a triangle.

Without loss of generality, we may assume that D is an interior point of ABC. Therefore, among the four circles which go through three of the points, only the circumcircle of ABC contains the fourth point. That is, $n_A = n_B = n_C = 0$ and $n_D = 1$.

Thus, in that case, we have $n_A + n_B + n_C + n_D = 1$.

Case 2. The convex hull is a quadrilateral, say $ABCD$.

Let $\alpha = \angle DAB$, $\beta = \angle ABC$, $\gamma = \angle BCD$, and $\delta = \angle CDA$. Then $\alpha + \beta + \gamma + \delta = 2\pi$. Since the four points are not concylic, it follows that either $\alpha + \gamma > \pi$ or $\beta + \delta > \pi$, but not both.

Without loss of generality, we may assume that $\alpha + \gamma > \pi$. Thus, $\angle BCD = \gamma > \pi - \alpha = \angle BAD$, which means that C is an interior point of the circumcircle of ABD. Hence, we have $n_C = 1$.

Similarly, we have $n_A = 1$. Since $\beta + \delta < \pi$, similar reasoning shows that $n_B = n_D = 0$. Thus, in that case, we have $n_A + n_B + n_C + n_D = 2$.

From above, we deduce that for each group of four points in S, say A, B, C, D, we have

$$n_A + n_B + n_C + n_D \;\leq\; 2\,, \tag{1}$$

with equality if and only if A, B, C, D are the vertices of a convex quadrilateral.

There are $\binom{n}{4}$ groups of four points in S. Summing over these groups the inequalities of the form (1), it follows that $m(S) \leq 2\binom{n}{4}$, with equality if and only if each group of four points in S is the set of vertices of a convex quadrilateral. From the lemma, it follows that $m(S) \leq 2\binom{n}{4}$, with equality if and only if the points of S are the vertices of a convex polygon. Thus, letting $f(n) = 2\binom{n}{4}$ proves the claim.

4. [7 points] Let x and y be any two real numbers. Prove that

$$3(x + y + 1)^2 + 1 \;\geq\; 3xy\,.$$

Under what conditions does equality hold?

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Houda Anoun, Bordeaux, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Amengual Covas.

For any real numbers X and Y,

$$X^2 + Y^2 + XY \;=\; \left(X + \tfrac{1}{2}Y\right)^2 + \tfrac{3}{4}Y^2 \;\geq\; 0\,,$$

and equality holds if and only if $X = Y = 0$.

Let x and y be any two real numbers. Letting $X = x + \frac{2}{3}$ and $Y = y + \frac{2}{3}$ above, we obtain

$$\left(x + \tfrac{2}{3}\right)^2 + \left(y + \tfrac{2}{3}\right)^2 + \left(x + \tfrac{2}{3}\right)\left(y + \tfrac{2}{3}\right) \;\geq\; 0\,.$$

Expanding and multiplying by 3 gives

$$3x^2 + 3y^2 + 3xy + 6x + 6y + 4 \;\geq\; 0\,.$$

This may be written as

$$3(x + y + 1)^2 - 3xy + 1 \;\geq\; 0\,,$$

from which we arrive at the desired inequality.

Equality holds if and only if $x + \frac{2}{3} = y + \frac{2}{3} = 0$; that is, if and only if $x = y = -\frac{2}{3}$.

6. Mr. Leonardo invited a group of children to go for a ride around a lake on his boat, in several turns. He later realized that the following things had happened:

- In each turn, there had been exactly three children on the boat.
- Each pair of children had been together on the boat in exactly one turn.

(a) [2 points] Prove that if Mr. Leonardo invited n children, then n must be a number of the form $6t + 1$ or $6t + 3$, where t is a non-negative integer.

(b) [5 points] Prove that, for any non-negative integer t, Mr. Leonardo can invite $6t + 3$ children under the above conditions.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

(a) There are $\frac{1}{2}n(n - 1)$ pairs of children. Since each turn concerns exactly three children, these pairs divide into k groups of three, where k is the number of turns. Thus, $\frac{1}{2}n(n-1) = 3k$, or $n(n-1) = 6k \equiv 0 \pmod{6}$.

Let A be one of the children. Since the remaining $n - 1$ children divide into disjoint pairs to take a turn with A, it follows that $n \equiv 1 \pmod{6}$, $n \equiv 3 \pmod{6}$, or $n \equiv 5 \pmod{6}$. However, if $n \equiv 5 \pmod{6}$, then $n(n - 1) \equiv 2 \not\equiv 0 \pmod{6}$. Hence, $n \equiv 1 \pmod{6}$ or $n \equiv 3 \pmod{6}$.

(b) Let $n = 6t + 3$, where t is a non-negative integer. If $t = 0$, only one turn suffices. Thus, we assume that $t \geq 1$.

Note that $n = 3(2t + 1)$, where $2t + 1 \geq 3$ is odd. Thus, we may identify the children with couples (x, r), where $x \in \{1, 2, \ldots, 2t + 1\}$ and $r \in \{0, 1, 2\}$. In the following, the computations are assumed to be modulo $2t + 1$ for the first component and modulo 3 for the second one.

Since $2t+1 \geq 3$ is odd, the number 2 has an inverse modulo $2t+1$; let $\frac{1}{2}$ denote that inverse. Then, the turns are formed by the following triples:

- all the triples of the form $\{(x,0), (x,1), (x,2)\}$;
- all the triples of the form $\left\{(x,r),(y,r),\left(\frac{1}{2}(x+y), r+1\right)\right\}$, where $x \not\equiv y \pmod{2t+1}$.

Remark. Triples satisfying the statement of the problem are said to form a *Steiner Triple System* (STS). From (a), a set with cardinality n has a STS only if $n = 1 \pmod 6$ or $n = 3 \pmod 6$. And from (b), the converse is true for $n = 3 \pmod 6$. In fact, the converse also holds for $n = 1 \pmod 6$, as first proved by Kirkman (1847).

References:

[1] T.P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. J.*, 2 (1847), pp. 191–204.

[2] J.H. van Lint, R.M. Wilson, *A course in Combinatorics*, second edition, Cambridge University Press, p. 236.

Lastly, we look at solutions from our readers to problems of the 53rd Polish Mathematical Olympiad 2001–2002, Final Round [2005 : 151–152].

1. Determine all triples of positive integers a, b, c such that a^2+1 and b^2+1 are prime numbers satisfying $(a^2+1)(b^2+1) = c^2+1$.

Solution by Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON.

The only solution is $(a, b, c) = (1, 2, 3)$.

Suppose that a, b, c satisfy

$$(a^2+1)(b^2+1) \;=\; c^2+1\,, \tag{1}$$

where $a^2+1 = p$ and $b^2+1 = q$ are both prime. If $p = q$, then from (1) we obtain $p^2 = c^2+1$, or $(p-c)(p+c) = 1$, which is clearly impossible. Thus, $p \neq q$. Without loss of generality, we may assume $p < q$. Then $a < b < c$.

Rewriting (1) as $a^2b^2 + a^2 + b^2 = c^2$, we have

$$(c-a)(c+a) \;=\; c^2 - a^2 \;=\; b^2(a^2+1) \;=\; b^2 p \tag{2}$$

$$\text{and}\quad (c-b)(c+b) \;=\; c^2 - b^2 \;=\; a^2(b^2+1) \;=\; a^2 q\,. \tag{3}$$

Since p and q are primes, we deduce from (2) and (3) that $p \mid (c-a)$ or $p \mid (c+a)$, and $q \mid (c-b)$ or $q \mid (c+b)$. Thus, we have four possible cases:

Case 1. $p \mid (c-a)$ and $q \mid (c-b)$.

Then $pq \mid (c-a)(c-b)$. Hence, $(c^2+1) \mid (c-a)(c-b)$, which is impossible, since $0 < (c-a)(c-b) < c^2 < c^2+1$.

Case 2. $p \mid (c+a)$ and $q \mid (c-b)$.

Then $pq \mid (c+a)(c-b)$. Hence, $(c^2+1) \mid (c+a)(c-b)$, which is impossible, since

$$c^2+1-(c+a)(c-b) \;=\; bc-ac+ab+1 \;=\; c(b-a)+ab+1 \;>\; 0\,.$$

Case 3. $p \mid (c-a)$ and $q \mid (c+b)$.

Then $(c^2+1) \mid (c-a)(c+b)$. Hence, $(c-a)(c+b) = k(c^2+1)$ for some $k \in \mathbb{N}$. However, $(c-a)(c+b) < c(2c) = 2c^2 < 2(c^2+1)$; whence, $k=1$. Therefore, $(c-a)(c+b) = c^2+1$, which can be rewritten as $(b-a)(c-a) = a^2+1 = p$.

Since $b-a < c-a$, we must have $b-a = 1$ and $c-a = p$. Since $b = a+1$, we see that a and b have opposite parity; thus, $p = a^2+1$ and $q = b^2+1$ must also have opposite parity. Since $p < q$, we conclude that $p = 2$. It follows that $a = 1$ and $b = 2$, and we obtain the solution $(a,b,c) = (1,2,3)$.

Case 4. $p \mid (c+a)$ and $q \mid (c+b)$.

Then $(c^2+1) \mid (c+a)(c+b)$. Hence, $(c+a)(c+b) = m(c^2+1)$ for some $m \in \mathbb{N}$. However,

$$\begin{aligned}(c+a)(c+b) &< (c+a)(c+b)+(c-a)(c-b)+(a-b)^2+a^2b^2\\ &= 2c^2+a^2+b^2+a^2b^2\\ &= 3c^2 < 3(c^2+1)\,,\end{aligned}$$

where we have used (1) in the second-last step. Thus, we see that $m = 2$. Then, from $(c+a)(c+b) = 2(c^2+1)$, we obtain

$$(a+b)c+ab \;=\; c^2+2\,. \tag{4}$$

If $p \neq 2$, then both p and q are odd, which implies that a and b are both even. Hence, c is also even. Let $a = 2a_1$, $b = 2b_1$, and $c = 2c_1$. Then (4) becomes $4(a_1+b_1)c_1+4a_1b_1 = 4c_1^2+2$ or $2(a_1+b_1)c_1+2a_1b_1 = 2c_1^2+1$, which is clearly impossible. Hence, $p = 2$ and $a = 1$. Substituting into (3), we then have $(c-b)(c+b) = q$. Hence, $c-b = 1$ and $c+b = q$. Solving, we have $b = (q-1)/2 = b^2/2$, from which it follows that $b = 2$, and again we are led to the solution $(a,b,c) = (1,2,3)$.

This completes the proof.

2. On sides AC and BC of an acute-angled triangle ABC, rectangles $ACPQ$ and $BKCL$ are erected outwardly. Assuming that these rectangles have equal areas, show that the vertex C, the circumcentre of triangle ABC, and the mid-point of segment PL are collinear.

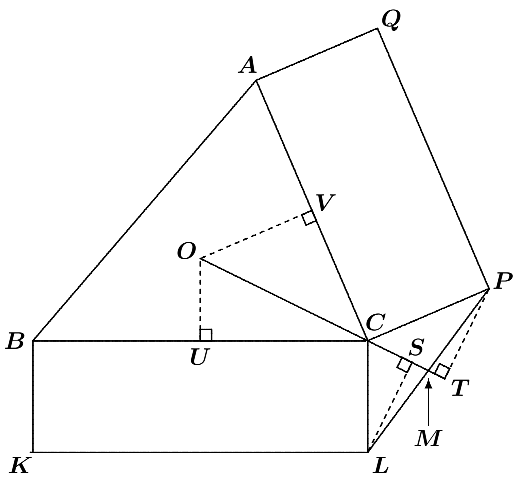
Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA. We first give Amengual Covas' write-up.

Let O be the circumcentre of $\triangle ABC$. Let OC (produced) intersect PL at M. We will prove that M is the mid-point of PL. The desired conclusion follows from this.

Let S and T be the feet of the perpendiculars to OC from L and P, respectively, and let U and V be the feet of the perpendiculars from O to BC and CA, respectively. Since $\angle OCU$ is a complementary angle for both $\angle LCS$ and $\angle COU$, we have $\angle LCS = \angle COU$. It follows that the right triangles OUC and CSL are similar. Hence, $LS/UC = CL/OC$. Similarly, $PT/VC = CP/OC$. Therefore,

$$LS \;=\; \frac{UC\cdot CL}{OC} \;=\; \frac{\frac{1}{2}[BKLC]}{OC} \;=\; \frac{\frac{1}{2}[ACPQ]}{OC} \;=\; \frac{VC\cdot CP}{OC} \;=\; PT\,.$$

This implies that $LM = MP$; that is M is the mid-point of PL.



Next we give the write-up by Bataille using complex numbers.

We consider the figure to be drawn in the complex plane with origin at the circumcentre O of $\triangle ABC$. We denote by M the mid-point of PL and, generally, by x the complex representation of the point X (so that $m = \frac{1}{2}(p+\ell)$, for example).

Now $CL\cdot CB = CP\cdot CA$, since rectangles $BCLK$ and $ACPQ$ have the same area. Therefore, we may define

$$\varrho \;=\; \frac{|\ell - c|}{|a-c|} \;=\; \frac{|p-c|}{|b-c|}\,.$$

Without loss of generality, we may suppose that triangle ABC is positively oriented. Letting $\gamma = \angle\overrightarrow{ACB}$, we have $\angle\overrightarrow{ACL} = \gamma + \frac{\pi}{2}$ and

$\angle\overrightarrow{BCP} = -(\gamma + \frac{\pi}{2})$. It follows that

$$\frac{\ell - c}{a - c} = \varrho e^{i(\gamma+\pi/2)} = i\varrho e^{i\gamma}$$

and

$$\frac{p - c}{b - c} = \varrho e^{-i(\gamma+\pi/2)} = -i\varrho e^{-i\gamma}\,.$$

Now,

$$\begin{aligned} m - c &= \tfrac{1}{2}(p + \ell) - c = \tfrac{1}{2}\big[(\ell - c) + (p - c)\big] \\ &= \tfrac{1}{2}\left[(a - c)\varrho i e^{i\gamma} - (b - c)\varrho i e^{-i\gamma}\right] \\ &= \tfrac{1}{2}\left[\varrho i(a e^{i\gamma} - b e^{-i\gamma}) + 2c\varrho \sin\gamma\right]\,. \end{aligned}$$

Since $\angle\overrightarrow{AOB} = 2\gamma$ and $OA = OB$, we get $b = ae^{2i\gamma}$, or $be^{-i\gamma} = ae^{i\gamma}$. Thus, $m - c = (\varrho\sin\gamma)c$, which means that $\overrightarrow{CM} = (\varrho\sin\gamma)\overrightarrow{OC}$. Thus, O, C, and M are collinear.

4. Prove that, for every integer $n \geq 3$ and every sequence of positive numbers $x_1, x_2, \ldots, x_n$, at least one of the following two inequalities is satisfied:

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}} \geq \frac{n}{2}\,, \qquad \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + x_{i-2}} \geq \frac{n}{2}\,.$$

(Note: Here $x_{n+1} = x_1$, $x_{n+2} = x_2$, $x_0 = x_n$, and $x_{-1} = x_{n-1}$.)

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. We give the solution of Díaz-Barrero, modified by the editor.

Denote the sums in the inequalities above by S_1 and S_2, respectively. If $S_1 < n/2$ and $S_2 < n/2$, then $S_1 + S_2 < n$. Therefore, it will be sufficient to prove that $S_1 + S_2 \geq n$.

We start with the inequality $\alpha + \dfrac{1}{\alpha} \geq 2$, which is true for all positive real numbers α. For each $i \in \{1, 2, \ldots, n\}$, let $\alpha_i = \frac{x_{i-1} + x_i}{x_i + x_{i+1}}$. Then $\alpha_i + \dfrac{1}{\alpha_i} \geq 2$ for each i, and hence,

$$2n \leq \sum_{i=1}^{n} \left(\alpha_i + \frac{1}{\alpha_i}\right) = \sum_{i=1}^{n} \left(\alpha_i + \frac{1}{\alpha_{i+1}}\right)\,,$$

where $\sum\limits_{i=1}^{n} \dfrac{1}{\alpha_i} = \sum\limits_{i=1}^{n} \dfrac{1}{\alpha_{i+1}}$ because the sum is cyclic. Thus,

$$\begin{aligned} 2n &\leq \sum_{i=1}^{n} \left(\frac{x_{i-1} + x_i}{x_i + x_{i+1}} + \frac{x_{i+1} + x_{i+2}}{x_i + x_{i+1}}\right) \\ &= \sum_{i=1}^{n} \left(\frac{x_i + x_{i+1}}{x_i + x_{i+1}} + \frac{x_{i-1} + x_{i+2}}{x_i + x_{i+1}}\right) = n + \sum_{i=1}^{n} \frac{x_{i-1} + x_{i+2}}{x_i + x_{i+1}}\,. \end{aligned}$$

Then

$$\begin{aligned} n &\leq \sum_{i=1}^{n} \frac{x_{i-1}}{x_i + x_{i+1}} + \sum_{i=1}^{n} \frac{x_{i+2}}{x_i + x_{i+1}} \\ &= \sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}} + \sum_{i=1}^{n} \frac{x_i}{x_{i-2} + x_{i-1}} \;=\; S_1 + S_2 \,, \end{aligned}$$

where we have again made use of the cyclic nature of the sums to shift the index.

6. Let k be a fixed positive integer. The infinite sequence $\{a_n\}$ is defined by the formulae $a_1 = k + 1$ and $a_{n+1} = a_n^2 - ka_n + k$ for $n \geq 1$. Show that if $m \neq n$, then the numbers a_m and a_n are relatively prime.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let $P(x) = x^2 - kx + k$. Then $P(a_n) = a_{n+1}$ for all integers $n \geq 1$. If $k \neq 4$, then the quadratic polynomial $P(x)$ has no integer root and it follows that $a_n \neq 0$ for all n. If $k = 4$, then $P(x) = (x - 2)^2$. Since $a_1 = 5$, it follows by induction that $a_n \geq 5$ for all n, and again we have $a_n \neq 0$ for all n.

Now, let $n \geq 1$ be given. Clearly, we have $a_{n+1} \equiv k \pmod{a_n}$. If $a_{n+t} \equiv k \pmod{a_n}$ for some $t > 0$, then

$$a_{n+t+1} \;\equiv\; k^2 - k^2 + k \;\equiv\; k \pmod{a_n} \,.$$

It follows by induction that $a_p \equiv k \pmod{a_q}$ for all integers p and q such that $p > q$.

Suppose there exist integers p and q such that $p > q \geq 1$ and $\gcd(a_p, a_q) > 1$. Let q be minimal among all such pairs. Let r be a prime which divides both a_p and a_q. Then r divides k, because $a_p \equiv k \pmod{a_q}$. We must have $q > 1$, because r cannot divide both k and $a_1 = k + 1$. Then r divides a_{q-1}^2, from the inductive construction of the sequence, so that a_q and a_{q-1} are not relatively prime. This contradicts the minimality of q, and we are done.

That completes this number of the *Corner*. The back-log is now cleared up and we are publishing reader solutions about one year after the problems appear; so please send me your nice solutions and generalizations, along with Olympiad Contests!