SKOLIAD No. 95

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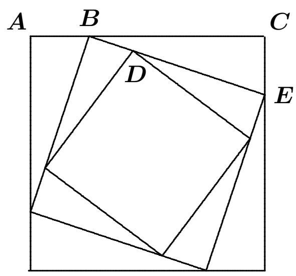
Please send your solutions to the problems in this edition by ***1 March, 2007***. A copy of **MATHEMATICAL MAYHEM Vol. 5** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our items for this issue come from the 5th annual CNU Regional Mathematics Contest. Only a selection of the problems has been included. Thanks go to R. Porsky, Christopher Newport University (CNU), Newport News, VA. [Some of these questions also appear on the 2004 Calgary Junior Mathematics Contest.]

5th Annual CNU Regional High School Mathematics Contest
Saturday November 13, 2004

1. Mr. Smith pours a full cup of coffee and drinks $\frac{1}{2}$ of it, deciding it is too strong and needs some milk. So he fills the cup with milk, stirs it, and tastes again, drinking another $\frac{1}{4}$ cup. Once again he fills the cup with milk, stirs it, and finds that this is just as he likes it. What ratio $\frac{\text{amount of coffee}}{\text{amount of milk}}$ does Mr. Smith like?

2. You have three inscribed squares, with the corners of each inner square at the $\frac{1}{4}$ point along the sides of its outer square. (Thus, for example, $AB = \frac{1}{4}AC$ and $BD = \frac{1}{4}BE$.) The area of the largest square is 64 cm^2. What is the area of the smallest square?



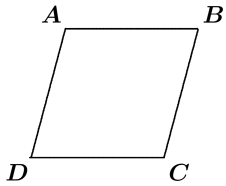
3. Solve the equation $\cos 2x = \cos x$ for $0 \leq x < 2\pi$.

4. The centre of a circle of radius 1 cm is on the circumference of a circle of radius 3 cm. How far (in cm) from the centre of the big circle do the common tangents of the two circles meet?

5. One root of $2hx^2 + (3h-6)x - 9 = 0$ is the negative of the other. Find the value of h.

6. Solve the equation $\sqrt{16x+1} - 2\sqrt[4]{16x+1} = 3$.

7. In the figure $ABCD$, all four sides have length 10 and the area is 60. What is the length of the shorter diagonal AC?



8. A man has 1000 equilateral triangular pieces of mosaic, all of side length 1 cm. He constructs the largest possible mosaic in the form of an equilateral triangle.

(a) What is the side length of the mosaic?

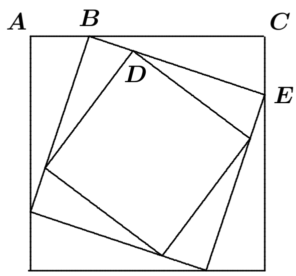
(b) How many pieces will he have left over?

5^{e} Concours Annuel CNU Régional de Mathématique du Secondaire

Samedi, le 13 Novembre 2004

1. M. Smith se verse une tasse pleine de café et en boit la moitié. Décidant que le café est trop fort, il remplit la tasse de lait, mélange le contenu et goûte de nouveau, buvant de nouveau $\frac{1}{4}$ de tasse. Il remplit de nouveau la tasse de lait, mélange le contenu et trouve le café alors à son goût. Quel ratio $\frac{\text{quantité de café}}{\text{quantité de lait}}$ est au goût de M. Smith?

2. On a 3 carrés inscrits, de telle sorte que les sommets d'un carré intérieur sont au $\frac{1}{4}$ des côtés de son carré extérieur (par exemple, $AB = \frac{1}{4}AC$ et $BD = \frac{1}{4}BE$). L'aire du carré le plus grand est 64 cm^2. Quelle est l'aire du plus petit?



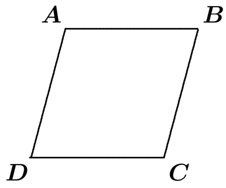
3. Résoudre l'équation $\cos 2x = \cos x$ pour $0 \leq x < 2\pi$.

4. Le centre d'un cercle de rayon 1 cm est sur la circonférence d'un cercle de rayon 3 cm. À quelle distance du centre du gros cercle se trouve le point d'intersection des tangentes communes des 2 cercles?

5. Une racine de $2hx^2 + (3h - 6)x - 9 = 0$ est l'opposé de l'autre. Trouver la valeur de h.

6. Résoudre l'équation $\sqrt{16x+1} - 2\sqrt[4]{16x+1} = 3$.

7. Les quatre côtés de la figure $ABCD$ mesurent 10 et son aire vaut 60. Quelle est la longueur de la diagonale la plus courte AC?



8. Un homme a 1000 triangles équilatéraux de côté 1 cm. Il décide de construire avec eux une mosaïque ayant pour forme le plus grand triangle équilatéral possible.

(a) Quel est le côté de cette mosaïque?

(b) Combien de morceaux lui reste-t-il?

Next we give solutions to the 2005 Maritime Mathematics Competition [2005 : 482–483].

2005 Maritime Mathematics Competition
March 3, 2005

1. A gardener owns a riding lawn mower and a push mower. It takes her 3 hours to cut the entire lawn with the push mower but only 75 minutes with the riding mower. One particular day, she cuts a portion of the lawn with the push mower and the rest with the riding lawn mower. If the total time to mow the lawn was 96 minutes, what fraction of the lawn was cut with the riding mower?

Official solution.

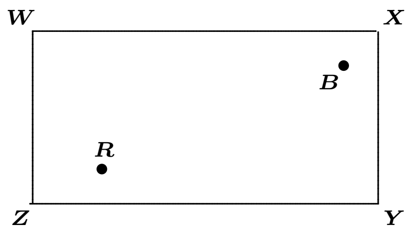
Suppose that the gardener used the riding mower for x minutes and the push mower for $96 - x$ minutes. The fraction of the lawn cut by the riding mower is then $x/75$, and the fraction cut by the push mower is $(96-x)/180$. Now we have

$$\frac{x}{75} + \frac{96-x}{180} = 1,$$

which gives $x = 60$. The fraction of the lawn cut by the riding mower is then $60/75 = 4/5$.

One incorrect solution was received.

2. Les bandes (côtés intérieurs) d'une table de billard forment le rectangle $WXYZ$ comme dans le diagramme ci-dessous. La bande WZ a cinq pieds de long. La bande WX a dix pieds de long. Une boule rouge (R) est située à un pied de YZ et à deux pieds de WZ. Une boule bleue (B) est située à un pied de WX et à un pied de XY. Nous voulons frapper la boule bleue pour qu'elle frappe la bande YZ, avec l'angle d'incidence égal à l'angle de réflexion, pour ensuite frapper la boule rouge. Quel point sur la bande YZ devons-nous viser ?



Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Modélisons la situation par une fonction valeur absolue. Soit y_1 la partie décroissante de la fonction et y_2 sa partie croissante. Par symétrie, on a $y_1 = -mx + b$ et $y_2 = mx - b$.

En R, on a : $y_1 = -mx + b$ qui donne $1 = -2m + b$.

En B, on a : $y_2 = mx - b$ qui donne $4 = 9m - b$.

On résoud ce système pour obtenir $m = \frac{5}{7}$ et $b = \frac{17}{7}$. L'abscisse à l'origine est le point que l'on recherche et il vaut $x = \frac{17}{5} = 3,4$.

3. Trois étudiants jouent un jeu où le perdant de chaque partie doit doubler l'argent de chacun des deux autres joueurs. Après trois parties, chaque joueur a perdu une fois et possède \$24. Combien d'argent possédait chaque étudiant au début du jeu ?

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Soit X, Y et Z les noms des trois joueurs. Alors, on utilisera x_0, x_1 et x_2 pour désigner les bourses au début de chaque partie pour X. Il en sera de même pour y_0, y_1 et y_2 et z_0, z_1 et z_2.

Soit Z ayant perdu la dernière partie, alors avant cette partie les montants valaient :

$$x_2 = \frac{24}{2} = 12\,, \qquad y_2 = \frac{24}{2} = 12\,, \qquad z_2 = 24 + x_2 + y_2 = 48\,.$$

Soit Y ayant perdue la deuxième partie, alors avant cette partie, les montants valaient :

$$x_1 = \frac{12}{2} = 6\,, \qquad z_1 = \frac{48}{2} = 24\,, \qquad y_1 = 12 + x_1 + z_1 = 42\,.$$

Soit X ayant perdue la première partie, alors avant cette partie, les montants valaient :

$$y_0 = \frac{42}{2} = 21\,, \qquad z_0 = \frac{24}{2} = 12\,, \qquad x_0 = 6 + y_0 + z_0 = 39\,.$$

Les montants initiaux sont donc \$39, \$21 et \$12.

4. Find all integers a for which the equation $x^3 - 13x + a = 0$ has three integer roots.

Official solution.

Let r_1, r_2, and r_3 be the roots of the equation. We may then write

$$\begin{aligned} x^3 - 13x + a &= (x - r_1)(x - r_2)(x - r_3) \\ &= x^3 - (r_1 + r_2 + r_3)x^2 \\ &\qquad +(r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 \,; \end{aligned}$$

equating coefficients on like powers of x gives

$$\begin{aligned} r_1 + r_2 + r_3 &= 0 \,, \\ r_1r_2 + r_1r_3 + r_2r_3 &= -13 \,, \\ -r_1r_2r_3 &= a \,. \end{aligned}$$

Now

$$r_1^2+r_2^2+r_3^2 = (r_1+r_2+r_3)^2-2(r_1r_2+r_1r_3+r_2r_3) = 0^2-2(-13) = 26 \,.$$

Since r_1, r_2, and r_3 are integers, their squares are also integers. We seek, therefore, three squares of integers that sum to 26. The only possibilities are $\{0,\ 1,\ 25\}$ and $\{1,\ 9,\ 16\}$. If $\{r_1^2,\ r_2^2,\ r_3^2\} = \{0,\ 1,\ 25\}$, then we have $\{r_1,\ r_2,\ r_3\} = \{0,\ \pm 1,\ \pm 5\}$, which is impossible since $r_1 + r_2 + r_3 = 0$. Therefore, we must have $\{r_1^2,\ r_2^2,\ r_3^2\} = \{1,\ 9,\ 16\}$. Hence, there are two possibilities for the three roots, namely $\{1,\ 3,\ -4\}$ and $\{-1,\ -3,\ 4\}$. Finally, there are two possible values for $a = -r_1r_2r_3$, namely 12 and -12.

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

5. Triangle ABC is right-angled at A. Let x and y denote the lengths of the sides AB and AC, respectively. Suppose that the point D on BC is such that $\angle DAC = 30^\circ$. Determine the length of AD in terms of x and y.

Official solution, modified by the editors.

Place a coordinate system on the triangle so that A is at the origin, B is at $(x, 0)$, and C is at $(0, y)$. The equation of the line through B and C is then $Y = (-y/x)X + y$. Since $\angle DAC = 30^\circ$, the line through A and D has slope $\cot 30^\circ = \sqrt{3}$, and its equation is $Y = \sqrt{3}\,X$. We now find the coordinates of D, the point of intersection of the two lines. We have $\sqrt{3}\,X = (-y/x)X + y$, which gives $X = \dfrac{xy}{y + x\sqrt{3}}$. Then $Y = \dfrac{xy\sqrt{3}}{y + x\sqrt{3}}$.

Now the length of AD is

$$\sqrt{\left(\frac{xy}{y + x\sqrt{3}}\right)^2 + \left(\frac{xy\sqrt{3}}{y + x\sqrt{3}}\right)^2} = \frac{2xy}{y + x\sqrt{3}} \,.$$

6. Evaluate the following sum.

$$\frac{1}{1^4+1^2+1}+\frac{2}{2^4+2^2+1}+\cdots+\frac{2005}{2005^4+2005^2+1}.$$

Official solution.

The general term in the series has the form

$$\begin{aligned}\frac{n}{n^4+n^2+1} &= \frac{n}{(n^2+1)^2-n^2} = \frac{n}{(n^2-n+1)(n^2+n+1)} \\ &= \frac{1}{2}\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right).\end{aligned}$$

Now $n^2+n+1=(n+1)^2-(n+1)+1$, which implies that the given sum may be written as a telescoping series as follows:

$$\begin{aligned}&\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{7}\right)+\frac{1}{2}\left(\frac{1}{7}-\frac{1}{13}\right)+\cdots \\ &\qquad +\frac{1}{2}\left(\frac{1}{2005^2-2005+1}-\frac{1}{2005^2+2005+1}\right) \\ &= \frac{1}{2}\left(1-\frac{1}{2005^2+2005+1}\right) = \frac{1}{2}\left(\frac{2005^2+2005}{2005^2+2005+1}\right) \\ &= \frac{2011015}{4022031}.\end{aligned}$$

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

That brings us to the end of another issue. This month's winner of a past Volume of Mayhem is Jean-François Désilets. Congratulations Jean-François! Continue sending in your contests and solutions.