

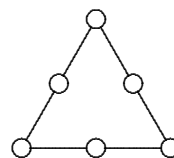
Pólya's Paragon

Magic Triangles: Beyond the Elementary Idea

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Puzzles commonly appear in mathematics texts as challenges. Usually the puzzles invite answers only, and the underlying mathematical principles are not discussed. The objective of this *Pólya's Paragon* is to unearth the mathematics beneath the surface of some seemingly elementary puzzles. This focus on the mathematical ideas makes the puzzles more advanced than they may initially appear. Specifically, the idea of magic triangles will be central to our discussion.

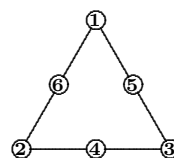
The basic magic triangle problem may be posed as follows: Consider the digits 1, 2, 3, 4, 5, and 6. Place each of the digits in one of the circles so that the sum of the digits along each side of the triangle is the same.



You are welcome to solve the problem; but the likely result is that you will find a suitable arrangement and stop. Technically it would be correct to say that you had successfully met the challenge. Mathematically there is much more than meets the eye. This particular problem has proven to be a rich teaching problem in my experiences with school age students and (prospective) teachers. Why? There is a sense of contentment among those who find a satisfactory solution. This sense is shaken by the realization that a neighbouring student has a different arrangement—in fact, a different sum. The arrangements are not merely reflections of one another but are different solutions to the same problem. Some people have never experienced such a moment in mathematics. They wonder “How many solutions are there?”, or “How will I know when they have all been found?” Here we examine the problem in greater detail to answer these questions.

Observe that there are three identical sums each made up of three distinct digits selected from 1, 2, 3, 4, 5, and 6. Three of these digits will be placed at the vertices of the triangle and will thus appear in two such sums, whereas the remaining digits will appear in only one sum. Since each of 1, 2, 3, 4, 5, and 6 appears in at least one sum, the total of the three sums must be greater than 21. How much greater? By exactly the sum of the three digits placed at the vertices.

Suppose that the numbers 1, 2, and 3 appear at the vertices. The total of the three sums would become $21 + 6 = 27$, making the sum along each side equal to 9. We will refer to this common sum (that is, 9 in this case) as the magic sum. Is it possible to make a triangle under such conditions? Indeed, the arrangement falls out automatically, as shown to the right.



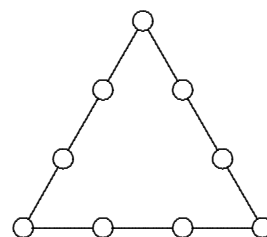
Now consider the other extreme possibility, where 4, 5, and 6 are placed at the vertices. Here the magic sum would be $(21 + 15)/3 = 12$. Again, a magic triangle falls out quickly with the placement of 3 between 4 and 5, 1 between 5 and 6, and 2 between 6 and 4.

We have found two solutions. Are there more? If so, they must have magic sums of 10 or 11. Consider the case of 10. Since the sum of the digits 1 through 6 is 21, the numbers at the vertices must total 9 (or $3 \times 10 - 21$). Two possible combinations of digits total 9: (2, 3, 4) and (1, 3, 5). The first combination will not produce a magic triangle because the 3 and 4 would need another 3 placed between them to obtain 10. The second combination does produce a magic triangle with the placement of 6 between 1 and 3, 2 between 3 and 5, and 4 between 5 and 1. We can apply similar reasoning to the case of 11 as a magic sum. Again we find that a solution exists.

In summary, there are four distinct magic sums possible, each of which corresponds to one solution. The elementary puzzle no longer appears like a five or ten minute challenge.

Now consider a larger triangle as shown at right. Place each of the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 in one of the circles so that the sum of the digits along each side of the triangle is the same.

This problem may seem, at first glance, to be the same as the first problem, but the analysis is more complicated now. Observing that the sum of the digits 1 through 9 is 45, we can quickly verify that the smallest possible magic sum is $(45 + 1 + 2 + 3)/3$ and the largest is $(45 + 7 + 8 + 9)/3$. That is, the magic sums potentially range from 17 to 23 inclusive. Here is where the challenge is handed over to you, the reader about to be turned solver.



1. Show that magic triangles can be found with magic sums of 17 and 23.
2. Prove that a magic triangle with magic sum 22 does not exist.
3. Find all magic sums between 17 and 22 that produce magic triangles.

A more familiar member of the family of magical figures is the magic square in which the sum of each of the rows, columns, and main diagonals is the same. The best known example appears at right. Its magic sum is 15 and the entry in its middle square is 5. Prove that the middle entry in any 3×3 magic square must equal one-third of the magic sum.

8	1	6
3	5	7
4	9	2

I will close this Paragon with a problem given to me by Gerry Rising at University of Buffalo. It is based upon a square configuration with another twist. The entries in each row and column of the figure shown to the right are in arithmetic progression. Determine the value represented by *.

			*	
	74			
				186
		103		
0				

Enjoy the challenges and know that you are welcome to send along comments on any of the problems or ideas discussed in this feature.