Problem of the Month

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It's time for a problem that is a bit more, shall we say, "complex".

Problem (1993 Fermat Contest)

If $i^2 = -1$ and $p^3 = 5 + \sqrt{2}i$ and $q^3 = 5 - \sqrt{2}i$, then there are three real values for $p + q$. One of these values is

(A) -5 (B) 10 (C) 2 (D) -2 (E) 5

Problems about complex numbers tend to appear very seldom on contests, at least in North America. Thus, there is some merit in doing a quick review of some concepts, which may or may not be useful.

If z is a complex number, then we can write $z = a + bi$ for real numbers a and b. The number a is called the *real part* of z (denoted $\mathfrak{Re}(z) = a$) and the number b is called the *imaginary part* of z (denoted $\mathfrak{Im}(z) = b$).

For $z = a + bi$, the *modulus* of z, denoted $|z|$, is the length of the vector in $\mathbb{R}^2$ induced by z; that is, $|z| = \sqrt{a^2 + b^2}$. Also, the *conjugate* of z is the complex number $\overline{z} = a - bi$.

Solution 1. For lack of any more clever idea, we start by letting $p = a + bi$ for some real numbers a and b. Then

$$\begin{aligned} p^3 &= (a+bi)^3 = a^3 + 3a^2bi + 3ab^2i^2 + b^3i^3 \\ &= a^3 + 3a^2bi - 3ab^2 - b^3i \qquad \text{(since } i^2 = -1\text{)} \\ &= (a^3 - 3ab^2) + i(3a^2b - b^3)\,. \end{aligned}$$

Since $p^3 = 5 + \sqrt{2}i$, then, by comparing real parts and imaginary parts, we obtain the system of equations

$$\begin{aligned} a^3 - 3ab^2 &= 5\,, \\ 3a^2b - b^3 &= \sqrt{2}\,. \end{aligned}$$

This gets us into one of these proverbial "good news, bad news" situations: the good news is that we have a system of two equations in two unknowns; the bad news is, just look at the system!

Can we simplify this somehow? Look at the modulus of p and p^3. From the given information,

$$|p^3| = |5 + \sqrt{2}i| = \sqrt{5^2 + (\sqrt{2})^2} = \sqrt{27} = 3\sqrt{3}\,.$$

Racking our brains to remember any useful connection, we might remember that $|p^3| = |p|^3$ (in general, $|p^n| = |p|^n$). Thus, $|p|$, which is real and non-negative, is the real non-negative cube root of $|p^3| = 3\sqrt{3}$, which is $\sqrt{3}$. Hence, $|a + bi| = \sqrt{3}$; that is, $a^2 + b^2 = 3$, or $b^2 = 3 - a^2$.

How does this help? Substituting this into the first equation in the system above gives us an equation in a only, namely $a^3 - 3a(3 - a^2) = 5$, or $4a^3 - 9a - 5 = 0$. We can see that $a = -1$ is a root; thus, we get $(a + 1)(4a^2 - 4a - 5) = 0$. The other two roots are going to be ugly, so we will focus on $a = -1$ for a minute. If $a = -1$, then $b^2 = 3 - a^2 = 2$, or $b = \pm\sqrt{2}$. If we check these back in the two equations in a and b above, we can see that $b = \sqrt{2}$ works. Thus, one value of p is $p = -1 + \sqrt{2}i$.

But we wanted a real value for $p+q$. In fact, since q^3 is the conjugate of p^3, then a possible value for q is the conjugate of p (since $(\overline{p})^3 = \overline{p}^3$—another long-lost fact about complex numbers). Thus, $q = -1 - \sqrt{2}i$ satisfies the required equation, and in this case $p + q = -2$, giving the answer (D).

Actually, we could have seen that $q = -1 - \sqrt{2}i$ gives the correct value for q^3 by noticing that, when we checked $b = -\sqrt{2}$ in the second equation, we got $-\sqrt{2}$ on the right side instead of $\sqrt{2}$. This would be exactly the equation we would have gotten were we focussing on q instead of p.

Now, we certainly have not found all possible values for $p+q$ or even all possible real values for $p+q$. We could have done this, though, by completely solving the cubic equation to get the remaining two possible values for a, finding the corresponding values of b, and finally sorting out the values of q. Possible, but not necessary here. (This is an advantage of doing multiple-choice questions—sometimes you can get out of doing the ugly work.)

Here is another solution, which looks a bit like something that a magician would pull out of a hat.

Solution 2 Since $p^3 = 5 + \sqrt{2}i$ and $q^3 = 5 - \sqrt{2}i$, then $p^3 + q^3 = 10$ (the imaginary parts cancel) and

$$p^3q^3 \;=\; (5 + \sqrt{2}i)(5 - \sqrt{2}i) \;=\; 5^2 + (\sqrt{2})^2 = 27\,.$$

(We might think to multiply these together, as they are conjugates, and multiplying conjugates often makes nice things happen.)

Since $p^3q^3 = 27$, then $(pq)^3 = 27$; whence, a possible value for pq is 3. (There are two other complex values of pq, but they are not so simple. With some luck, we may not have to worry about them. Let's see what we get with $pq = 3$.) We have a real value for $p^3 + q^3$ and we are seeking a real value for $p + q$. We might think to be sneaky and factor

$$p^3 + q^3 \;=\; (p + q)(p^2 - pq + q^2) \;=\; (p + q)\big((p + q)^2 - 3pq\big)\,.$$

We set $S = p + q$ and run with our $pq = 3$ value to see what happens. Substituting what we know, we get $10 = S\big(S^2 - 3(3)\big)$, or $S^3 - 9S - 10 = 0$.

We could use the Rational Roots Theorem (remember this?) to try to narrow down the list of possible rational roots, obtaining $S = -2$ as a root. We could then factor out $S + 2$ to get $(S + 2)(S^2 - 2S - 5) = 0$. But wait! We have what we want already: a real value for $p + q = S$, which is -2.

Now that was pretty clever, which of course means that I didn't think of it! Hopefully, I won't develop a "complex" about this.