Pólya's Paragon

It Ain't So Complex (Part 2)

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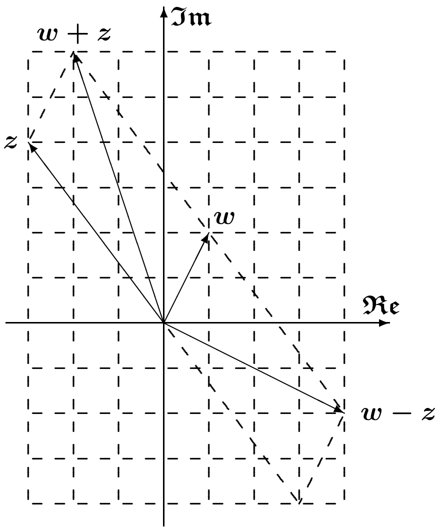
Last month, we started exploring the wonderful world of complex numbers. You were left with the assignment of looking at these numbers geometrically by assigning the complex number $z = a + bi$ to the point with co-ordinates (a, b).

Let us now try to find a geometric interpretation for addition and subtraction. Let $w = 1 + 2i$ and $z = -3 + 4i$. Then, as we saw last month,

$$\begin{aligned} w + z &= -2 + 6i \\ \text{and} \quad w - z &= 4 - 2i\,. \end{aligned}$$

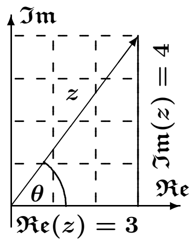


The complex numbers w, z, $w + z$, and $w - z$ are shown in the diagram to the right. Notice that $w + z$ is the diagonal of a parallelogram formed using w and z as sides.

Similarly, to interpret $w - z$, we just think of this subtraction as the addition $w + (-z)$, and note that $-z$ is represented by an arrow with the same length as z, but in the opposite direction.

(When you study *vectors*, you will recognize this geometric way of adding. In essence, we are treating the complex numbers as vectors.)

If you look at those arrows, it should become evident that we could define the arrow (complex number) by the co-ordinates of its tip (its real and imaginary parts) *or* by its length and direction. If we define the direction by the angle the arrow makes with the positive real axis, we have a second way to reference the complex number z.



For example, let $z = 3 + 4i$ (as shown in the diagram to the right). The length of z (called the *modulus* of z and denoted by $|z|$) is $\sqrt{3^2 + 4^2} = 5$. The angle θ (called the *argument* of z) satisfies $\tan\theta = \frac{4}{3}$.

Thus, we can express any complex number z in two forms: $z = a + bi$ (the *rectangular* form) and $z = r(\cos\theta + i\sin\theta)$ (the *polar* form).

Let us now go back and look at the multiplication example from last month:

$$(1 + 2i) \times (-3 + 4i) \;=\; -11 - 2i\,.$$

Putting each of these numbers in polar form, we get

$$\begin{aligned} w \;=\; 1 + 2i &\;=\; \sqrt{5}(\cos\theta_1 + i\sin\theta_1)\,, \\ z \;=\; -3 + 4i &\;=\; 5(\cos\theta_2 + i\sin\theta_2)\,, \\ w \times z \;=\; -11 - 2i &\;=\; 5\sqrt{5}(\cos\theta_3 + i\sin\theta_3)\,, \end{aligned}$$

where θ_1, θ_2, and θ_3 are the arguments of the three complex numbers.

Notice that the length of $w \times z$ is $|w \times z| = 5\sqrt{5}$, which is the product of $|w| = \sqrt{5}$ and $|z| = 5$. What about the argument? If you calculate the arguments θ_1, θ_2, θ_3, you will find that θ_3 is coterminal with $\theta_1 + \theta_2$. (Try it!) Thus, when we multiply complex numbers, we *add* their arguments. We will explore this idea and its implications next month.

For homework, you have a couple of tasks:

1. Work out a rule for dealing with division of complex numbers in polar form; that is, find how r_3 and θ_3 are related to r_1, r_2, θ_1 and θ_2, if

$$r_3(\cos\theta_3 + i\sin\theta_3) \;=\; [r_1(\cos\theta_1 + i\sin\theta_1)] \div [r_2(\cos\theta_2 + i\sin\theta_2)]\,.$$

2. Design another mathematical form for the expression $\cos\theta + i\sin\theta$ which suggests the rule for multiplication as a natural consequence. *Hint:* What function converts addition to multiplication?

3. Solve the equation $z^2 = i$.

Finally, from last month's homework, we were looking at generalizing division of complex numbers. If we look at $1 \div (c + di)$, we want a complex number, $x + yi$ such that $(x + yi)(c + di) = 1$. Following the method presented last month, you should get $x + yi = \dfrac{c - di}{c^2 + d^2}$. Thus, $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}$ (which you could see if you multiply the numerator and denominator by $\overline{z}$). This gives us a quicker method of computing division. Thus, the general division question gives us:

$$(a + bi) \div (c + di) \;=\; \frac{ac + bd}{c^2 + d^2} + \frac{-ad + bc}{c^2 + d^2}i\,.$$

Happy problem solving; see you next month.