Problem of the Month

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In last month's Problem of the Month, we looked for integer solutions to an equation. This month, we will look at a problem where we have to find all *real* solutions to a nasty-looking equation.

Problem (1991 Austrian Mathematical Olympiad, Round 2)
Determine all real numbers x which satisfy the equation

$$\frac{1}{x}+\frac{1}{x+2}-\frac{1}{x+4}-\frac{1}{x+6}-\frac{1}{x+8}-\frac{1}{x+10}+\frac{1}{x+12}+\frac{1}{x+14} \;=\; 0\,.$$

This equation does not look particularly easy to solve. (Can you see one solution by inspection, using some sort of symmetry?) One approach would be to multiply the equation by the product of the eight denominators on the left side. This would give, on the left side, eight terms with seven factors each, which could then be expanded and simplified to yield an enormous polynomial, which we could then attempt to factor. I would not recommend this approach.

Solution: First we will make a substitution to take advantage of the symmetry in the equation. The denominators form an arithmetic sequence x, $x+2$, $x+4$, $x+6$, $x+8$, $x+10$, $x+12$, $x+14$. Rewrite these in terms of the "middle" term (okay, there are eight terms, so there is no middle term, but we can use the average, $x+7$). By making the substitution $y=x+7$, these terms become $y-7$, $y-5$, $y-3$, $y-1$, $y+1$, $y+3$, $y+5$, $y+7$, and the equation becomes

$$\frac{1}{y-7}+\frac{1}{y-5}-\frac{1}{y-3}-\frac{1}{y-1}-\frac{1}{y+1}-\frac{1}{y+3}+\frac{1}{y+5}+\frac{1}{y+7} \;=\; 0\,.$$

This looks a bit more appealing and approachable. (You may have seen this idea in other places. For example, if you are trying to write an arithmetic sequence with five terms, it is sometimes more convenient to use $a-2d$, $a-d$, a, $a+d$, $a+2d$ instead of a, $a+d$, $a+2d$, $a+3d$, $a+4d$.)

At some point, we must start combining the fractions. But how shall we do this in a way that is as painless as possible? Well, we should try to use the symmetry. After fiddling around a bit, we might notice that

$$\frac{1}{y-c}+\frac{1}{y+c} \;=\; \frac{y+c+y-c}{(y-c)(y+c)} \;=\; \frac{2y}{y^2-c^2}\,.$$

Using this, we can regroup our terms and then simplify as follows:

$$\left(\frac{1}{y-7}+\frac{1}{y+7}\right)+\left(\frac{1}{y-5}+\frac{1}{y+5}\right)-\left(\frac{1}{y-3}+\frac{1}{y+3}\right)-\left(\frac{1}{y-1}+\frac{1}{y+1}\right)$$
$$= \frac{2y}{y^2-49}+\frac{2y}{y^2-25}-\frac{2y}{y^2-9}-\frac{2y}{y^2-1}\,.$$

Aha! All of the terms now have a common factor $2y$. Thus, either $y = 0$, or we can factor out $2y$ and obtain

$$\frac{1}{y^2-49} + \frac{1}{y^2-25} - \frac{1}{y^2-9} - \frac{1}{y^2-1} \;=\; 0\,.$$

What about the solution $y = 0$? This solution makes perfect sense. If we substitute $y = 0$ into our first equation involving y, the terms cancel each other in $+/-$ pairs. This solution also corresponds to $x = -7$, which of course yields the same type of cancellation back in the original equation. (This is the "symmetrical" solution at which I hinted earlier.)

Where to now? Again, we should probably combine pairs of the four terms that are left. But which pairs? Well, if we combine a "$+$" term with a "$-$" term, then the resulting combination will have no y^2 term in the numerator—watch! First, we will rearrange:

$$\begin{aligned}
\frac{1}{y^2-49} - \frac{1}{y^2-1} &= -\frac{1}{y^2-25} + \frac{1}{y^2-9}\,,\\
\frac{48}{(y^2-49)(y^2-1)} &= \frac{-16}{(y^2-9)(y^2-25)}\,,\\
3(y^2-9)(y^2-25) &= -(y^2-49)(y^2-1)\,,\\
4y^4 - 152y^2 + 724 &= 0\,,\\
y^4 - 38y^2 + 181 &= 0\,.
\end{aligned}$$

We have ended up with a quartic polynomial which is actually a quadratic polynomial in y^2. Hence, we can solve it using the quadratic formula to obtain $y^2 = 19 \pm 6\sqrt{5}$. Since these values for y^2 are both positive, we can continue by taking square roots to obtain four possible values of y, namely $y = \pm\sqrt{19 + 6\sqrt{5}}$ and $y = \pm\sqrt{19 - 6\sqrt{5}}$.

Now, we have five values for y which solve the equation. (We must not forget about our earlier discovery that $y = 0$ is a solution.) We need to translate these back into values of x, remembering that $x = y - 7$. Then we get the values of x which are solutions:

$$-7\,, \quad -7 \pm \sqrt{19 + 6\sqrt{5}}\,, \quad -7 \pm \sqrt{19 - 6\sqrt{5}}\,.$$

Normally at this stage, I would recommend that we go back and check these solutions by substituting into the original equation... Maybe this is one place where a calculator would come in handy!

We have solved a tough equation here. Along the way, we have used a lot of different techniques—making a substitution motivated by symmetry, combining terms in thoughtful ways, solving a special quartic polynomial by noticing that it is really a quadratic polynomial in disguise...the list goes on!