Mayhem Solutions

M141. *Proposed by the Mayhem Staff.*

Create a list of perfect squares in which all of the digits are perfect squares (that is, 0, 1, 4, 9).

Solution by Doug Newman, Lancaster, CA, USA, modified by the editor.

We want to create a list of perfect squares, n^2, whose digits come from the set $\{0, 1, 4, 9\}$. To make the task manageable, the number of values, n, to be checked must be pared down. The following are strategies to do this:

1. Only check n if the least significant digit is 0, 1, 2, 3, 7, 8, or 9. These will give an n^2 with least significant digit of 0, 1, 4, or 9.
2. Limit the ranges of values of n such that the most significant digits of n^2 are some combination of 0, 1, 4, and/or 9. (For example, since $\sqrt{1.4} = 1.1832\ldots$ and $\sqrt{1.5} = 1.2247\ldots$, if the most significant digits of n are between $11832\ldots$ and $12247\ldots$, then the most significant digits of n^2 will be $14\ldots$. Hence, when checking 3 digit numbers, you need only check 119, 120, 121, and 122 for candidates.)
3. When a value for n is rejected, then subsequent values of n that have the same leading digits as the rejected number are also rejected. For example, since 83^2 does not have the desired property, all n starting with 83 are rejected. [*Ed:* This may lead to missing some values. For example, since $139^2 = 19321$, we would reject 139; thus, any number starting with $139\ldots$ would be rejected, but $1393^2 = 1940449$, which has the desired property.]

Checking $1 \le n \le 10\,000$, we get the following (clearly if n has the desired property, then $10n$ also has the property; such numbers are not listed below; that is, $2^2 = 4$ is listed but $20^2 = 400$ is not listed, but is nevertheless considered part of the list):

n	n^2	n	n^2	n	n^2	n	n^2
1	1	21	441	138	19044	1002	1004004
2	4	38	1444	201	40401	2001	4004001
3	9	97	9409	212	44944	7001	49014001
7	49	102	10404	701	491401	9997	99949991
12	144	107	11449	997	994009		

From this point on, it is clear that patterns are present for the values of n

- 1002, 1020, 1200
- 2001, 2010, 2100
- 7001, 7010

- 9700, 9970, 9997

These can be used to get larger values for n^2.

[*Ed:* The patterns above can be proved quite easily. If $n = 10^a + 2\times 10^b$ with $a \neq b$, then $n^2 = 10^{2a} + 4 \times 10^{a+b} + 4 \times 10^{2b}$, which demonstrates that the first two patterns generalize. The third can be proved in a similar way. Note also that the solver did not find solutions in some of his ranges, but that does not mean that there are not others to be found. The problem remains open for other solutions.]

M142. *Proposed by Ali Feizmohammadi, University of Toronto, Toronto, ON.*

For every natural number n, define $S(n)$ to be the unique integer m (if it exists) which satisfies the equation

$$n \;=\; \lfloor m\rfloor + \left\lfloor \frac{m}{2!}\right\rfloor + \left\lfloor \frac{m}{3!}\right\rfloor + \cdots + \left\lfloor \frac{m}{k!}\right\rfloor + \ldots\,,$$

where $\lfloor x\rfloor$ is the greatest integer not exceeding x.

(a) Find $S(3438)$.

(b) Does there exist a number k such that, for any non-negative integer n, at least one of $S(n+1)$, $S(n+2)$, ..., $S(n+k)$ exists?

Solution by the proposer.

(a) First note that any integer m can be uniquely written in the form $m = a_1 \cdot 1! + a_2 \cdot 2! + \cdots + a_k \cdot k! + \ldots$, where $0 \le a_j \le j$. Now notice that

$$\sum_{j=1}^{\infty} \left\lfloor \frac{m}{j!}\right\rfloor \;=\; \sum_{j=1}^{\infty} \left\lfloor \sum_{i=1}^{\infty} \frac{a_i \cdot i!}{j!}\right\rfloor\,,$$

but

$$\left\lfloor \sum_{i=1}^{\infty} \frac{a_i \cdot i!}{j!}\right\rfloor \;=\; \left\lfloor \sum_{i=1}^{j-1} \frac{a_i \cdot i!}{j!} + \sum_{i=j}^{\infty} \frac{a_i \cdot i!}{j!}\right\rfloor\,.$$

Since

$$\sum_{i=1}^{j-1} \frac{a_i \cdot i!}{j!} \;\le\; \sum_{i=1}^{j-1} \frac{i \cdot i!}{j!} \;\le\; \frac{j!-1}{j!} \;<\; 1\,,$$

and since the second sum on the right consists only of integers, the first sum on the right contributes nothing to the value of the right side; thus,

$$\left\lfloor \sum_{i=1}^{\infty} \frac{a_i \cdot i!}{j!}\right\rfloor \;=\; \left\lfloor \sum_{i=j}^{\infty} \frac{a_i \cdot i!}{j!}\right\rfloor \;=\; a_j + a_{j+1}\cdot(j+1) + a_{j+2}\cdot(j+2)(j+1) + \cdots$$

Now let us consider $m + \left\lfloor \frac{m}{2!} \right\rfloor + \cdots = 3438$. Obviously, $m < 7!$; hence, $m = a \cdot 6! + b \cdot 5! + c \cdot 4! + d \cdot 3! + e \cdot 2 + f$. By the above argument, we have to solve for:

$$a(1 + 6 + 6 \times 5 + \cdots + 6!) + b(1 + 5 + 5 \times 4 + \cdots + 5!) + \cdots + f \;=\; 3438\,,$$

which is equivalent to

$$1237a + 206b + 41c + 10d + 3e + f \;=\; 3438\,.$$

One can easily verify that $a = 2$, $b = 4$, $c = 3$, $d = 1$, $e = 2$, $f = 1$ works. But the answer is unique (since $S(n)$ is a strictly increasing integer-valued function). Therefore, the solution is $m = 2003$.

(b) Obviously not! Let m be such that $S(m) = k! - 1$. Then

$$\lfloor k+1 \rfloor + \cdots + \left\lfloor \frac{k+1}{j!} \right\rfloor - \lfloor k \rfloor - \left\lfloor \frac{k}{2!} \right\rfloor - \cdots \;\geq\; k\,.$$

Taking k to be sufficiently large we can get blocks of consecutive integers of arbitrary length, none of which belong to range of S.

Also solved by Doug Newman, Lancaster, CA, USA.

M143. *Proposed by the Mayhem Staff.*

Find the equation(s) of the line(s) through the point $(2, 5)$ for which the y–intercept is a prime number and the x–intercept is an integer.

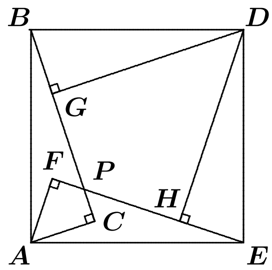
Solution by Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

Let k be the slope of a line through the point $(2, 5)$. The line then has equation $y - 5 = k(x - 2)$. To find the x-intercept, set $y = 0$, which yields $x = 2 - \frac{5}{k}$. Similarly, the y-intercept is given by $y = -2k + 5$. Since the intercepts are integers, we must have $k = \pm 1$ (with y-intercepts of 3 or 7), or $k = \pm\frac{1}{2}$ (with y-intercepts of 4 or 6), $k = \pm\frac{5}{2}$ (with y-intercepts of 0 or 10), or $k = \pm 5$ (with y-intercepts of -5 or 15). Since the only prime numbers among this list of y-intercepts are 3 and 7, we must have $k = \pm 1$, and the equations of the lines are $y - 5 = \pm(x - 2)$.

Also solved by Doug Newman, Lancaster, CA, USA.

M144. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A square $ABDE$ is drawn on the hypotenuse AB of right triangle ABC so that C lies in the interior of the square. A directly similar right triangle BDG is drawn so that G lies in the interior of the square. Indirectly similar right triangles EDH and AEF are drawn so that H and F lie in the interior of the square. Let BC and EF intersect at P. Determine the area of quadrilateral $DGPH$ in terms of the legs CA and CB of the original right triangle.



Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Since triangles ABC and AFE are congruent, triangles AFP and ACP are also congruent (since they are right-angled, $AF = AC$, and AP is a common hypotenuse). Then $\angle FAP = \angle PAC$. Since $\angle CAE = \angle FAB$, we have $\angle PAE = \angle PAB = 45°$. Hence, P lies over AD.

Triangles GPD and CPA are similar, because, from $GD \parallel AC$, we have $\angle GDP = \angle PAC$.

Therefore,

$$\frac{GD}{AC} = \frac{GP}{PC} = \frac{PD}{AP}$$

$$\text{and} \quad \frac{BC}{AC} = \frac{GP}{PC} = \frac{PD}{AP} \tag{1}$$

because $GD = BC$.

From (1) we have

$$GP = \frac{BC \cdot PC}{AC} \tag{2}$$

Since $BG = AC$, and $GP + PC = BC - AC$, from (2) we have

$$\frac{BC \cdot PC}{AC} + PC = BC - AC$$

$$PC = \frac{(BC - AC)AC}{BC + AC} \tag{3}$$

It is easy to prove that triangles GDP and DPH are congruent. Therefore, from (2) and (3) we get

$$[DGHP] = GD \cdot GP = BC \cdot GP = \frac{(BC - AC) \cdot BC^2}{BC + AC}.$$

($[DGHP]$ denotes the area of the quadrilateral $DGHP$.)

Also solved by Doug Newman, Lancaster, CA, USA.

M145. *Proposé par Ovidiu-Gabriel Dinu, Balcesti-Valcea, Roumanie.*

Trouver tous les nombres naturels n pour lesquels n, $n+2$, $n+6$, $n+8$ et $n+14$ sont premiers.

Solution par Houda Anoun, LaBri, Bordeaux, France.

Soit n un entier naturel, et soit $S = \{n, n+2, n+6, n+8, n+14\}$. C'est facile de vérifier que

$$\exists k \in S : \; k \equiv 0 \pmod 5 \; .$$

En effet, on a ce qui suit :

$$\begin{aligned} n &\equiv n \pmod 5 \\ n+2 &\equiv n+2 \pmod 5 \\ n+6 &\equiv n+1 \pmod 5 \\ n+8 &\equiv n+3 \pmod 5 \\ n+14 &\equiv n+4 \pmod 5 \; . \end{aligned}$$

Parce que n, $n+1$, $n+2$, $n+3$, $n+4$ sont 5 nombres consécutifs, il existe forcément un qui est divisible par 5.

Soit k le nombre divisible par 5 qui fait partie de S. On a $\forall i \in S$, i est premier donc k est lui aussi premier. Ainsi, $k = 5$.

D'autre part on a ce qui suit : $n + 14 > n + 8 > n + 6 > 5$. Il reste donc deux possibilités à savoir $k = n = 5$ ou $k = n + 2 = 5$. Le premier cas permet d'engendrer la liste suivante $\{5, 7, 11, 13, 19\}$ dont les éléments sont tous des nombres premiers, en revanche, le second cas engendre $\{3, 5, 9, 11, 17\}$ qui ne constitue pas une liste de nombre premiers.

Seule la première solution est retenue alors. Donc $n = 5$.

En outre résolu par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne ; Roger He, étudiant de la catégorie 10, Prince of Wales Collegiate, St. John's, NL ; et Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentine.