MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier mars 2006**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M207. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

A midi, Iphigénie quitte la maison pour faire une promenade à bicyclette, maintenant une moyenne de 20 km/h sur un sentier agréablement plat. Un peu plus tard, sa mère se rend compte qu'elle a oublié son pique-nique et envoie Electre le lui porter en vélo. Electre arrive à maintenir une vitesse de 30 km/h. Mais voilà que le ciel s'assombrit et que l'orage menace. Si bien qu'exactement une demi-heure après le départ de Electre, on envoie Oreste pour amener à ses deux soeurs de quoi se protéger contre la pluie. Oreste arrive à maintenir une vitesse de 40 km/h, si bien que les trois enfants, ayant suivi le même chemin, se rencontrent exactement au même moment. A quelle heure la rencontre a-t-elle eu lieu ?

M208. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Déterminer tous les triangles distincts ayant un côté de longueur 6, les deux autres côtés étant des entiers et le périmètre étant numériquement égal à la surface.

M209. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que $3x^2 + 4y^2$ et $4x^2 + 3y^2$ ne peuvent être simultanément des carrés parfaits pour tous les entiers positifs x et y.

M210. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Une grille 9×9 est subdivisée en neuf sous-grilles de 3×3, appelées boîtes. Chaque ligne et chaque colonne de la grille 9×9 de même que chaque boîte 3×3 doivent contenir les chiffres de 1 à 9.

Compléter la grille ci-contre.

4				9			8	
			5			7		
6	2	3	7				4	
	4	9					7	3
7	6					9	2	
	3				2	4	1	5
		2			6			
	1			5				7

M211. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Deux cercles de rayon r sont tangents extérieurement. Ils sont aussi intérieurement tangents aux côtés d'un triangle rectangle de côtés 3, 4 et 5, l' hypoténuse du triangle étant tangente aux deux cercles. Déterminer r.

M212. *Proposé par Robert Bilinski, Collège Montmorency, Laval, QC.*

Dans le programme d'ordinateur Excel, les colonnes sont indiquées par des lettres. Les 26 premières colonnes comportent les lettres de A à Z. La $27^{\text{ième}}$ colonne est intitulée AA ; la $28^{\text{ième}}$ colonne est intitulée AB.

(a) Quel est le numéro de la colonne intitulée DXA ?

(b) Quel est l'indication de la 2005-ième colonne ?

. . . .

M207. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

At noon, Iphigenia set off on a bike ride from her home in Saskatoon, maintaining a leisurely pace of 20 km/h on the pleasantly level terrain. Later, her mother noticed that she had forgotten her lunch, and sent Electra off on her bike to meet her; Electra maintained a steady pace of 30 km/h. But then the sky darkened and the storm clouds gathered. So, exactly a half hour after Electra left, Orestes was sent off to meet the others with rain gear. Orestes rode at a steady pace of 40 km/h. All three followed the same route. As it happened, the three siblings met at exactly the same time. What time was that?

M208. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine all distinct triangles having one side of length 6, with the other two sides being integers, and the perimeter numerically equal to the area.

M209. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that $3x^2 + 4y^2$ and $4x^2 + 3y^2$ cannot be simultaneously perfect squares for all x, y positive integers.

M210. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A 9×9 grid is subdivided into nine 3×3 smaller grids, called boxes. Each row and each column of the 9×9 grid, and each 3×3 box, must contain each of the digits 1 through 9.

Complete the grid on the right.

4				9			8	
			5			7		
6	2	3	7				4	
	4	9					7	3
7	6					9	2	
	3				2	4	1	5
		2			6			
	1			5				7

M211. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Two circles of radius r are externally tangent. They are also internally tangent to the sides of a right triangle of sides 3, 4, and 5, with the hypotenuse of the triangle being tangent to both circles. Determine r.

M212. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

In the computer program Excel, the columns are labelled with letters. The first 26 columns are labelled with the letters A to Z. The 27^{th} column is labelled AA; the 28^{th} column is labelled AB.

(a) What is the number of the column labelled DXA?

(b) What label appears on the 2005^{th} column?

Mayhem Solutions

M141. *Proposed by the Mayhem Staff.*

Create a list of perfect squares in which all of the digits are perfect squares (that is, 0, 1, 4, 9).

Solution by Doug Newman, Lancaster, CA, USA, modified by the editor.

We want to create a list of perfect squares, n^2, whose digits come from the set $\{0, 1, 4, 9\}$. To make the task manageable, the number of values, n, to be checked must be pared down. The following are strategies to do this:

1. Only check n if the least significant digit is 0, 1, 2, 3, 7, 8, or 9. These will give an n^2 with least significant digit of 0, 1, 4, or 9.

2. Limit the ranges of values of n such that the most significant digits of n^2 are some combination of 0, 1, 4, and/or 9. (For example, since $\sqrt{1.4} = 1.1832\ldots$ and $\sqrt{1.5} = 1.2247\ldots$, if the most significant digits of n are between $11832\ldots$ and $12247\ldots$, then the most significant digits of n^2 will be $14\ldots$. Hence, when checking 3 digit numbers, you need only check 119, 120, 121, and 122 for candidates.)

3. When a value for n is rejected, then subsequent values of n that have the same leading digits as the rejected number are also rejected. For example, since 83^2 does not have the desired property, all n starting with 83 are rejected. [*Ed:* This may lead to missing some values. For example, since $139^2 = 19321$, we would reject 139; thus, any number starting with $139\ldots$ would be rejected, but $1393^2 = 1940449$, which has the desired property.]

Checking $1 \leq n \leq 10\,000$, we get the following (clearly if n has the desired property, then $10n$ also has the property; such numbers are not listed below; that is, $2^2 = 4$ is listed but $20^2 = 400$ is not listed, but is nevertheless considered part of the list):

n	n^2	n	n^2	n	n^2	n	n^2
1	1	21	441	138	19044	1002	1004004
2	4	38	1444	201	40401	2001	4004001
3	9	97	9409	212	44944	7001	49014001
7	49	102	10404	701	491401	9997	99949991
12	144	107	11449	997	994009		

From this point on, it is clear that patterns are present for the values of n

- 1002, 1020, 1200
- 2001, 2010, 2100
- 7001, 7010
- 9700, 9970, 9997

These can be used to get larger values for n^2.

[*Ed:* The patterns above can be proved quite easily. If $n = 10^a + 2 \times 10^b$ with $a \neq b$, then $n^2 = 10^{2a} + 4 \times 10^{a+b} + 4 \times 10^{2b}$, which demonstrates that the first two patterns generalize. The third can be proved in a similar way. Note also that the solver did not find solutions in some of his ranges, but that does not mean that there are not others to be found. The problem remains open for other solutions.]

M142. *Proposed by Ali Feizmohammadi, University of Toronto, Toronto, ON.*

For every natural number n, define $S(n)$ to be the unique integer m (if it exists) which satisfies the equation

$$n = \lfloor m \rfloor + \left\lfloor \frac{m}{2!} \right\rfloor + \left\lfloor \frac{m}{3!} \right\rfloor + \cdots + \left\lfloor \frac{m}{k!} \right\rfloor + \cdots,$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(a) Find $S(3438)$.

(b) Does there exist a number k such that, for any non-negative integer n, at least one of $S(n+1)$, $S(n+2)$, $\ldots$, $S(n+k)$ exists?

Solution by the proposer.

(a) First note that any integer m can be uniquely written in the form $m = a_1 \cdot 1! + a_2 \cdot 2! + \cdots + a_k \cdot k! + \ldots$, where $0 \le a_j \le j$. Now notice that

$$\sum_{j=1}^{\infty} \left\lfloor \frac{m}{j!} \right\rfloor = \sum_{j=1}^{\infty} \left\lfloor \sum_{i=1}^{\infty} \frac{a_i \cdot i!}{j!} \right\rfloor ,$$

but

$$\left\lfloor \sum_{i=1}^{\infty} \frac{a_i \cdot i!}{j!} \right\rfloor = \left\lfloor \sum_{i=1}^{j-1} \frac{a_i \cdot i!}{j!} + \sum_{i=j}^{\infty} \frac{a_i \cdot i!}{j!} \right\rfloor .$$

Since

$$\sum_{i=1}^{j-1} \frac{a_i \cdot i!}{j!} \le \sum_{i=1}^{j-1} \frac{i \cdot i!}{j!} \le \frac{j! - 1}{j!} < 1 ,$$

and since the second sum on the right consists only of integers, the first sum on the right contributes nothing to the value of the right side; thus,

$$\left\lfloor \sum_{i=1}^{\infty} \frac{a_i \cdot i!}{j!} \right\rfloor = \left\lfloor \sum_{i=j}^{\infty} \frac{a_i \cdot i!}{j!} \right\rfloor = a_j + a_{j+1} \cdot (j+1) + a_{j+2} \cdot (j+2)(j+1) + \cdots$$

Now let us consider $m + \left\lfloor \frac{m}{2!} \right\rfloor + \cdots = 3438$. Obviously, $m < 7!$; hence, $m = a \cdot 6! + b \cdot 5! + c \cdot 4! + d \cdot 3! + e \cdot 2 + f$. By the above argument, we have to solve for:

$$a(1 + 6 + 6 \times 5 + \cdots + 6!) + b(1 + 5 + 5 \times 4 + \cdots + 5!) + \cdots + f = 3438 ,$$

which is equivalent to

$$1237a + 206b + 41c + 10d + 3e + f = 3438 .$$

One can easily verify that $a = 2$, $b = 4$, $c = 3$, $d = 1$, $e = 2$, $f = 1$ works. But the answer is unique (since $S(n)$ is a strictly increasing integer-valued function). Therefore, the solution is $m = 2003$.

(b) Obviously not! Let m be such that $S(m) = k! - 1$. Then

$$\lfloor k + 1 \rfloor + \cdots + \left\lfloor \frac{k+1}{j!} \right\rfloor - \lfloor k \rfloor - \left\lfloor \frac{k}{2!} \right\rfloor - \cdots \ge k .$$

Taking k to be sufficiently large we can get blocks of consecutive integers of arbitrary length, none of which belong to range of S.

Also solved by Doug Newman, Lancaster, CA, USA.

M143. *Proposed by the Mayhem Staff.*

Find the equation(s) of the line(s) through the point $(2, 5)$ for which the y–intercept is a prime number and the x–intercept is an integer.

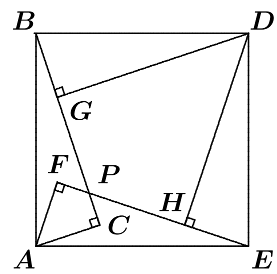
Solution by Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

Let k be the slope of a line through the point $(2,5)$. The line then has equation $y-5=k(x-2)$. To find the x-intercept, set $y=0$, which yields $x=2-\frac{5}{k}$. Similarly, the y-intercept is given by $y=-2k+5$. Since the intercepts are integers, we must have $k=\pm 1$ (with y-intercepts of 3 or 7), or $k=\pm\frac{1}{2}$ (with y-intercepts of 4 or 6), $k=\pm\frac{5}{2}$ (with y-intercepts of 0 or 10), or $k=\pm 5$ (with y-intercepts of -5 or 15). Since the only prime numbers among this list of y-intercepts are 3 and 7, we must have $k=\pm 1$, and the equations of the lines are $y-5=\pm(x-2)$.

Also solved by Doug Newman, Lancaster, CA, USA.

M144. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A square $ABDE$ is drawn on the hypotenuse AB of right triangle ABC so that C lies in the interior of the square. A directly similar right triangle BDG is drawn so that G lies in the interior of the square. Indirectly similar right triangles EDH and AEF are drawn so that H and F lie in the interior of the square. Let BC and EF intersect at P. Determine the area of quadrilateral $DGPH$ in terms of the legs CA and CB of the original right triangle.



Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Since triangles ABC and AFE are congruent, triangles AFP and ACP are also congruent (since they are right-angled, $AF=AC$, and AP is a common hypotenuse). Then $\angle FAP=\angle PAC$. Since $\angle CAE=\angle FAB$, we have $\angle PAE=\angle PAB=45^\circ$. Hence, P lies over AD.

Triangles GPD and CPA are similar, because, from $GD \parallel AC$, we have $\angle GDP=\angle PAC$.

Therefore,

$$\frac{GD}{AC}=\frac{GP}{PC}=\frac{PD}{AP}$$

$$\text{and}\quad \frac{BC}{AC}=\frac{GP}{PC}=\frac{PD}{AP} \tag{1}$$

because $GD=BC$.

From (1) we have

$$GP=\frac{BC\cdot PC}{AC} \tag{2}$$

Since $BG=AC$, and $GP+PC=BC-AC$, from (2) we have

$$\frac{BC\cdot PC}{AC}+PC=BC-AC$$

$$PC = \frac{(BC - AC)AC}{BC + AC} \qquad (3)$$

It is easy to prove that triangles GDP and DPH are congruent. Therefore, from (2) and (3) we get

$$[DGHP] = GD \cdot GP = BC \cdot GP = \frac{(BC - AC) \cdot BC^2}{BC + AC} .$$

($[DGHP]$ denotes the area of the quadrilateral $DGHP$.)

Also solved by Doug Newman, Lancaster, CA, USA.

M145. *Proposé par Ovidiu-Gabriel Dinu, Balcesti-Valcea, Roumanie.*

Trouver tous les nombres naturels n pour lesquels n, $n+2$, $n+6$, $n+8$ et $n + 14$ sont premiers.

Solution par Houda Anoun, LaBri, Bordeaux, France.

Soit n un entier naturel, et soit $S = \{n, n + 2, n + 6, n + 8, n + 14\}$. C'est facile de vérifier que

$$\exists k \in S : \ k \equiv 0 \pmod 5 .$$

En effet, on a ce qui suit :

$$\begin{aligned} n &\equiv n \pmod 5 \\ n+2 &\equiv n+2 \pmod 5 \\ n+6 &\equiv n+1 \pmod 5 \\ n+8 &\equiv n+3 \pmod 5 \\ n+14 &\equiv n+4 \pmod 5 . \end{aligned}$$

Parce que n, $n + 1$, $n + 2$, $n + 3$, $n + 4$ sont 5 nombres consécutifs, il existe forcément un qui est divisible par 5.

Soit k le nombre divisible par 5 qui fait partie de S. On a $\forall i \in S$, i est premier donc k est lui aussi premier. Ainsi, $k = 5$.

D'autre part on a ce qui suit : $n + 14 > n + 8 > n + 6 > 5$. Il reste donc deux possibilités à savoir $k = n = 5$ ou $k = n + 2 = 5$. Le premier cas permet d'engendrer la liste suivante $\{5, 7, 11, 13, 19\}$ dont les éléments sont tous des nombres premiers, en revanche, le second cas engendre $\{3, 5, 9, 11, 17\}$ qui ne constitue pas une liste de nombre premiers.

Seule la première solution est retenue alors. Donc $n = 5$.

En outre résolu par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne ; Roger He, étudiant de la catégorie 10, Prince of Wales Collegiate, St. John's, NL ; et Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentine.

Problem of the Month

Ian VanderBurgh, University of Waterloo

In last month's Problem of the Month, we looked for integer solutions to an equation. This month, we will look at a problem where we have to find all *real* solutions to a nasty-looking equation.

Problem (1991 Austrian Mathematical Olympiad, Round 2)
Determine all real numbers x which satisfy the equation

$$\frac{1}{x}+\frac{1}{x+2}-\frac{1}{x+4}-\frac{1}{x+6}-\frac{1}{x+8}-\frac{1}{x+10}+\frac{1}{x+12}+\frac{1}{x+14} \;=\; 0\,.$$

This equation does not look particularly easy to solve. (Can you see one solution by inspection, using some sort of symmetry?) One approach would be to multiply the equation by the product of the eight denominators on the left side. This would give, on the left side, eight terms with seven factors each, which could then be expanded and simplified to yield an enormous polynomial, which we could then attempt to factor. I would not recommend this approach.

Solution: First we will make a substitution to take advantage of the symmetry in the equation. The denominators form an arithmetic sequence x, $x+2$, $x+4$, $x+6$, $x+8$, $x+10$, $x+12$, $x+14$. Rewrite these in terms of the "middle" term (okay, there are eight terms, so there is no middle term, but we can use the average, $x+7$). By making the substitution $y=x+7$, these terms become $y-7$, $y-5$, $y-3$, $y-1$, $y+1$, $y+3$, $y+5$, $y+7$, and the equation becomes

$$\frac{1}{y-7}+\frac{1}{y-5}-\frac{1}{y-3}-\frac{1}{y-1}-\frac{1}{y+1}-\frac{1}{y+3}+\frac{1}{y+5}+\frac{1}{y+7} \;=\; 0\,.$$

This looks a bit more appealing and approachable. (You may have seen this idea in other places. For example, if you are trying to write an arithmetic sequence with five terms, it is sometimes more convenient to use $a-2d$, $a-d$, a, $a+d$, $a+2d$ instead of a, $a+d$, $a+2d$, $a+3d$, $a+4d$.)

At some point, we must start combining the fractions. But how shall we do this in a way that is as painless as possible? Well, we should try to use the symmetry. After fiddling around a bit, we might notice that

$$\frac{1}{y-c}+\frac{1}{y+c} \;=\; \frac{y+c+y-c}{(y-c)(y+c)} \;=\; \frac{2y}{y^2-c^2}\,.$$

Using this, we can regroup our terms and then simplify as follows:

$$\left(\frac{1}{y-7}+\frac{1}{y+7}\right)+\left(\frac{1}{y-5}+\frac{1}{y+5}\right)-\left(\frac{1}{y-3}+\frac{1}{y+3}\right)-\left(\frac{1}{y-1}+\frac{1}{y+1}\right)$$
$$= \;\; \frac{2y}{y^2-49}+\frac{2y}{y^2-25}-\frac{2y}{y^2-9}-\frac{2y}{y^2-1}\,.$$

Aha! All of the terms now have a common factor $2y$. Thus, either $y = 0$, or we can factor out $2y$ and obtain

$$\frac{1}{y^2-49} + \frac{1}{y^2-25} - \frac{1}{y^2-9} - \frac{1}{y^2-1} = 0 .$$

What about the solution $y = 0$? This solution makes perfect sense. If we substitute $y = 0$ into our first equation involving y, the terms cancel each other in $+/-$ pairs. This solution also corresponds to $x = -7$, which of course yields the same type of cancellation back in the original equation. (This is the "symmetrical" solution at which I hinted earlier.)

Where to now? Again, we should probably combine pairs of the four terms that are left. But which pairs? Well, if we combine a "$+$" term with a "$-$" term, then the resulting combination will have no y^2 term in the numerator—watch! First, we will rearrange:

$$\begin{aligned}
\frac{1}{y^2-49} - \frac{1}{y^2-1} &= -\frac{1}{y^2-25} + \frac{1}{y^2-9}\,, \\
\frac{48}{(y^2-49)(y^2-1)} &= \frac{-16}{(y^2-9)(y^2-25)}\,, \\
3(y^2-9)(y^2-25) &= -(y^2-49)(y^2-1)\,, \\
4y^4 - 152y^2 + 724 &= 0\,, \\
y^4 - 38y^2 + 181 &= 0\,.
\end{aligned}$$

We have ended up with a quartic polynomial which is actually a quadratic polynomial in y^2. Hence, we can solve it using the quadratic formula to obtain $y^2 = 19 \pm 6\sqrt{5}$. Since these values for y^2 are both positive, we can continue by taking square roots to obtain four possible values of y, namely $y = \pm\sqrt{19+6\sqrt{5}}$ and $y = \pm\sqrt{19-6\sqrt{5}}$.

Now, we have five values for y which solve the equation. (We must not forget about our earlier discovery that $y = 0$ is a solution.) We need to translate these back into values of x, remembering that $x = y - 7$. Then we get the values of x which are solutions:

$$-7\,, \quad -7 \pm \sqrt{19+6\sqrt{5}}\,, \quad -7 \pm \sqrt{19-6\sqrt{5}}\,.$$

Normally at this stage, I would recommend that we go back and check these solutions by substituting into the original equation... Maybe this is one place where a calculator would come in handy!

We have solved a tough equation here. Along the way, we have used a lot of different techniques—making a substitution motivated by symmetry, combining terms in thoughtful ways, solving a special quartic polynomial by noticing that it is really a quadratic polynomial in disguise...the list goes on!

Pólya's Paragon

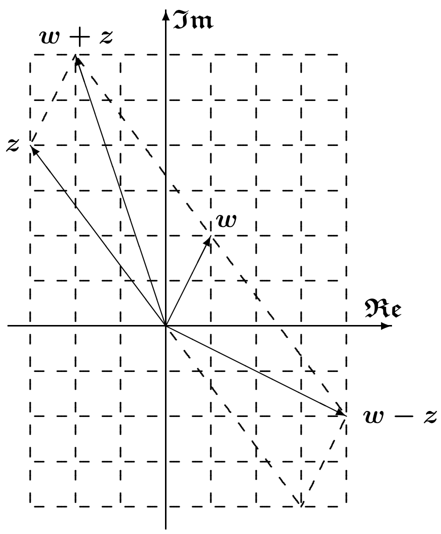
It Ain't So Complex (Part 2)

Shawn Godin

Last month, we started exploring the wonderful world of complex numbers. You were left with the assignment of looking at these numbers geometrically by assigning the complex number $z = a + bi$ to the point with co-ordinates (a, b).

Let us now try to find a geometric interpretation for addition and subtraction. Let $w = 1 + 2i$ and $z = -3 + 4i$. Then, as we saw last month,

$$\begin{aligned} w + z &= -2 + 6i \\ \text{and} \quad w - z &= 4 - 2i\,. \end{aligned}$$

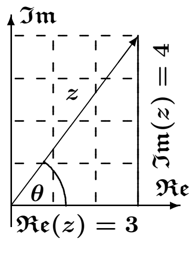


The complex numbers w, z, $w + z$, and $w - z$ are shown in the diagram to the right. Notice that $w + z$ is the diagonal of a parallelogram formed using w and z as sides.

Similarly, to interpret $w - z$, we just think of this subtraction as the addition $w + (-z)$, and note that $-z$ is represented by an arrow with the same length as z, but in the opposite direction.

(When you study *vectors*, you will recognize this geometric way of adding. In essence, we are treating the complex numbers as vectors.)

If you look at those arrows, it should become evident that we could define the arrow (complex number) by the co-ordinates of its tip (its real and imaginary parts) *or* by its length and direction. If we define the direction by the angle the arrow makes with the positive real axis, we have a second way to reference the complex number z.



For example, let $z = 3 + 4i$ (as shown in the diagram to the right). The length of z (called the *modulus* of z and denoted by $|z|$) is $\sqrt{3^2 + 4^2} = 5$. The angle θ (called the *argument* of z) satisfies $\tan\theta = \frac{4}{3}$.

Thus, we can express any complex number z in two forms: $z = a + bi$ (the *rectangular* form) and $z = r(\cos\theta + i\sin\theta)$ (the *polar* form).

Let us now go back and look at the multiplication example from last month:

$$(1+2i) \times (-3+4i) \;=\; -11-2i\,.$$

Putting each of these numbers in polar form, we get

$$\begin{aligned} w \;=\; 1+2i &\;=\; \sqrt{5}(\cos\theta_1 + i\sin\theta_1)\,, \\ z \;=\; -3+4i &\;=\; 5(\cos\theta_2 + i\sin\theta_2)\,, \\ w\times z \;=\; -11-2i &\;=\; 5\sqrt{5}(\cos\theta_3 + i\sin\theta_3)\,, \end{aligned}$$

where θ_1, θ_2, and θ_3 are the arguments of the three complex numbers.

Notice that the length of $w \times z$ is $|w \times z| = 5\sqrt{5}$, which is the product of $|w| = \sqrt{5}$ and $|z| = 5$. What about the argument? If you calculate the arguments θ_1, θ_2, θ_3, you will find that θ_3 is coterminal with $\theta_1 + \theta_2$. (Try it!) Thus, when we multiply complex numbers, we *add* their arguments. We will explore this idea and its implications next month.

For homework, you have a couple of tasks:

1. Work out a rule for dealing with division of complex numbers in polar form; that is, find how r_3 and θ_3 are related to r_1, r_2, θ_1 and θ_2, if

$$r_3(\cos\theta_3 + i\sin\theta_3) \;=\; [r_1(\cos\theta_1 + i\sin\theta_1)] \div [r_2(\cos\theta_2 + i\sin\theta_2)]\,.$$

2. Design another mathematical form for the expression $\cos\theta + i\sin\theta$ which suggests the rule for multiplication as a natural consequence. *Hint:* What function converts addition to multiplication?

3. Solve the equation $z^2 = i$.

Finally, from last month's homework, we were looking at generalizing division of complex numbers. If we look at $1 \div (c+di)$, we want a complex number, $x + yi$ such that $(x+yi)(c+di) = 1$. Following the method presented last month, you should get $x + yi = \dfrac{c-di}{c^2+d^2}$. Thus, $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}$ (which you could see if you multiply the numerator and denominator by $\overline{z}$). This gives us a quicker method of computing division. Thus, the general division question gives us:

$$(a+bi) \div (c+di) \;=\; \frac{ac+bd}{c^2+d^2} + \frac{-ad+bc}{c^2+d^2}i\,.$$

Happy problem solving; see you next month.