

## SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

**2958.** [2004 : 297, 300] Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.

Let  $ABC$  be a right triangle with right angle at  $A$ . Let  $ACDE$ ,  $BAFG$ , and  $CBIJ$  be squares mounted externally on the sides of  $\triangle ABC$ . Let  $H$  be the intersection of the interior angle bisector of angle  $A$  (extended) with the line segment  $EF$ , and let  $A'$  be the point outside the square  $CBIJ$  such that  $\triangle A'JI$  is directly congruent to  $\triangle ABC$ .

Show that  $A'DHG$  is a cyclic quadrilateral.

*Composite solution extracted from the solutions of several solvers marked with \* below.*

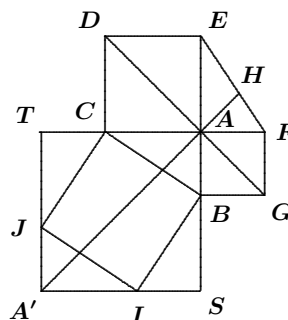
Complete the square  $ATA'S$  as shown, and join  $DAG$ . Clearly,  $DAG$  and  $HAA'$  are straight lines. Note that

$$EA = AC = TJ = A'I = SB = b,$$

and

$$AF = AB = SI = A'J = TC = c.$$

The length of the angle bisector  $AH$  is well known to be  $\frac{2bc}{b+c} \cos\left(\frac{A}{2}\right) = \frac{\sqrt{2}bc}{b+c}$ . It is easy to see that  $AA' = \sqrt{2}(b+c)$ ,  $DA = \sqrt{2}b$  and  $AG = \sqrt{2}c$ . Thus,  $DA \cdot AG = A'A \cdot AH$ , proving that  $HDA'G$  is a cyclic quadrilateral.



Solved by \*MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; \*FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; \*KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; \*TIMOTHY DILEO, student, California State University, Fullerton, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; \*TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; \*ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; \*MIHAELA VÂJJIAC, Chapman University, Orange, CA, USA, and BOGDAN SUCEAVĂ, California State University, Fullerton, CA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; \*YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; \*LI ZHOU, Polk Community College, Winter Haven, FL, USA; \*TITU ZVONARU, Comănești, Romania; and the proposer.

Other solvers used Ptolemy's Theorem, trigonometry, vectors, and/or coordinates.

**2959.** [2004 : 297, 300] Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Given a non-isosceles triangle  $ABC$ , prove that there exists a unique inscribed equilateral triangle  $PQR$  of minimal area, with  $P, Q, R$  on  $BC, CA$ , and  $AB$ , respectively. Construct it by straightedge and compass.

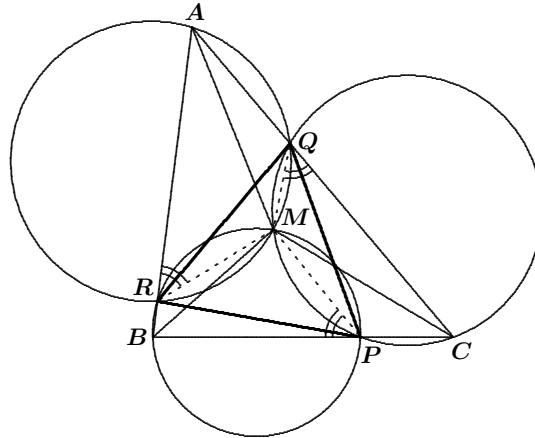
*Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Let  $PQR$  be an equilateral triangle inscribed in triangle  $ABC$ . Then, by Miquel's Theorem, the circumcircles of triangles  $QAR, RBP$ , and  $PCQ$  intersect in the Miquel point  $M$ . It is well-known that

$$\angle BMC = \angle CAB + \angle RPQ = A + 60^\circ, \quad (1)$$

$$\angle CMA = \angle ABC + \angle PQR = B + 60^\circ, \quad (2)$$

$$\angle AMB = \angle BCA + \angle QRP = C + 60^\circ. \quad (3)$$



Thus, for any equilateral triangle  $PQR$  inscribed in triangle  $ABC$ , the point  $M$  is characterized as the point that views the sides of triangle  $ABC$  under the above fixed angles. Furthermore, since the quadrilaterals  $MQAR, MRBP$ , and  $MPCQ$  are cyclic,

$$\angle MRA = \angle MPB = \angle MQC.$$

Hence, if we rotate the lines  $MP, MQ$ , and  $MR$  about point  $M$  at the same angle, then their intersections with the sides of  $\triangle ABC$  still form an equilateral triangle, because the rotation does not affect equations (1)–(3). In order to obtain a triangle of minimum area (which, for an equilateral triangle, translates into minimum perimeter as well), we need to minimize the distances  $MP, MQ$ , and  $MR$ . This is clearly achieved when the three angles  $MRA, MPB$ , and  $MQC$  are  $90^\circ$ ; that is, when  $\triangle PQR$  is the (unique) *pedal triangle* of the point  $M$ .

To construct it by straightedge and compass, first we construct the Miquel point as the intersection of two sets: the first one is the set of points

viewing the side  $BC$  under angle  $A + 60^\circ$ , and the second is the set of points viewing the side  $CA$  under angle  $B + 60^\circ$ . It remains to find the feet of the perpendiculars from point  $M$  to the sides of  $\triangle ABC$ . Clearly, both steps can be performed by straightedge and compass.

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Specht mentioned several published problems related to the construction at hand: Problems 623 and 624c [1981 : 116] and Problem 5 from the 1999 Brazilian Mathematical Olympiad, published in the book *Mathematical Olympiads: Problems and Solutions From Around the World 1999–2000*, by T. Andreescu and Z. Fang, MAA, 2002, p. 28.

**2964.** [2004 : 367, 370] Proposed by Joe Howard, Portales, NM, USA. (Inspired by Problem 80.D, Math. Gazette 80 (489) (1996) p. 606.)

Let  $x \in (0, \frac{\pi}{2})$ . Show that:

$$(a) \left[ \frac{2 + \cos x}{3} \right] \left[ \frac{2(1 - \cos x)}{x^2} \right] > \frac{1 + \cos x}{2};$$

$$(b) \frac{2 + \cos x}{3} < \sqrt{\frac{1 + \cos x}{2}} < \frac{2(1 - \cos x)}{x^2}.$$

*Solution by Michel Bataille, Rouen, France.*

(a) We will prove that the proposed inequality actually holds for all  $x \in (0, \pi)$ .

Let  $x \in (0, \pi)$ . The inequality can be rewritten as

$$\frac{1 + 2 \cos^2 \left( \frac{x}{2} \right)}{3} \cdot \frac{4 \sin^2 \left( \frac{x}{2} \right)}{x^2} > \cos^2 \left( \frac{x}{2} \right),$$

or  $\tan^2 u + 2 \sin^2 u > 3u^2,$  (1)

where  $u = x/2 \in (0, \pi/2)$ .

In order to prove (1), we first observe that,

$$3 \sin^2 u + u^4 > 3u^2, \quad \text{for } u > 0. \quad (2)$$

Indeed, for  $u > 0$ , the function  $f(u) = 3 \sin^2 u + u^4 - 3u^2$  is increasing, since its derivative is  $f'(u) = 3(\sin(2u) - 2u + \frac{(2u)^3}{3!}) > 0$ . Since  $f(0) = 0$ , we see that  $f(u) > 0$  for  $u > 0$ .

Similarly, by considering the function  $g(u) = \sin u \tan u - u^2$ , we find that

$$\sin u \tan u > u^2, \quad \text{for } u \in \left(0, \frac{\pi}{2}\right). \quad (3)$$

Indeed, we have  $g(0) = 0$  and, for  $u \in (0, \pi/2)$ ,

$$\begin{aligned} g'(u) &= \sin u \sec^2 u + \cos u \tan u - 2u \\ &= (\tan u) \left( \frac{1}{\cos u} + \cos u \right) - 2u \geq 2(\tan u - u) > 0, \end{aligned}$$

since  $\frac{1}{\cos u} + \cos u \geq 2$  and  $\tan u > u$ .



Draw  $K = AC \cap BD$ . We now construct the line  $PAQ$  parallel to  $BD$  as follows: Pick any point  $E$  on line  $AB$ . Construct  $G = EK \cap AD$  and  $F = ED \cap BG$ . From Ceva's Theorem applied to  $\triangle EBD$ , we have

$$\frac{EA}{AB} \cdot \frac{BK}{KD} \cdot \frac{DF}{FE} = 1,$$

which implies that  $EA/AB = EF/FD$ . Hence,  $FA$  is parallel to  $DB$ . Next, define  $P = FA \cap CB$  and  $Q = AF \cap CD$ , and draw  $R = AB \cap PD$  and  $S = AD \cap BQ$ . Then  $R$  and  $S$  are the centres of the parallelograms  $APBD$  and  $AQDB$ . Draw  $X = KR \cap AP$  and  $T = SX \cap KA$ . Then  $T$  is the centroid of  $\triangle KXY$ , where  $Y$  is the point where  $KS$  meets  $AQ$  (but  $Y$  need not be drawn); thus,  $AT = AK/3$ . Finally, draw  $M = DT \cap AB$ . By Menelaus's Theorem applied to  $\triangle ABK$  and transversal  $MD$ , we have

$$\frac{AM}{MB} = \frac{AT}{TK} \cdot \frac{KD}{BD} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

We conclude that  $AM = AB/5$ , as required.

*Conclusion:* Counting  $AC$ ,  $BD$ ,  $EK$ ,  $ED$ ,  $BG$ ,  $AF$ ,  $DP$ ,  $BQ$ ,  $KR$ ,  $SX$ ,  $DT$ , we found  $AB/5$  by drawing 11 lines.

## II. Comment by Victor Pambuccian, Phoenix, AZ.

The theory of straightedge constructions was worked out by Jacob Steiner (*Geometrische Konstruktionen, ausgeführt mittels der geraden Linie und eines festen Kreises*, Berlin, 1833). His solution to our problem (more precisely, to the problem of dividing a given segment into  $n$  equal parts by straightedge given a line parallel to it) was reproduced in A. Adler, *Theorie der geometrischen Konstruktionen*, Leipzig, 1906, pages 76-77. See <http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=05310001&seq=90&frames=0&view=50>.

Also solved by MICHEL BATAILLE, Rouen, France; TOSHIO SEIMIYA, Kawasaki, Japan; SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM-SOLVING GROUP; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

**2966.** [2004 : 368, 370] Proposed by Mikhael Kotchetov, Memorial University of Newfoundland, St. John's, NL.

Consider non-intersecting and non-congruent circles  $\Gamma_1$  and  $\Gamma_2$  with centres  $O_1$  and  $O_2$ , respectively. Let  $Q$  be the point of intersection of the two common tangents,  $t_1$  and  $t_2$ , which do not intersect the line segment  $O_1O_2$ . A common tangent,  $t_c$ , which intersects the segment  $O_1O_2$  meets the tangents  $t_1$  and  $t_2$  at  $E_1$  and  $E_2$ , respectively.

Let  $P$  be the mid-point of the line segment  $O_1O_2$ . Prove that  $P$ ,  $Q$ ,  $E_1$ , and  $E_2$  are concyclic.

I. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Suppose that  $\Gamma_1$  is smaller than  $\Gamma_2$ . Since  $O_1E_1$  and  $O_2E_1$  are internal and external bisectors of  $\angle QE_1E_2$ ,  $O_1E_1 \perp O_2E_1$ . Likewise,  $O_1E_2 \perp O_2E_2$ . Hence,  $O_1, E_1, O_2, E_2$  lie on a circle whose diameter is  $O_1O_2$ ; thus, its centre is  $P$ . Thus,  $\angle QPE_2 = \angle O_1PE_2 = 2\angle O_1E_1E_2 = \angle QE_1E_2$ , completing the proof.

II. *Solution by Michel Bataille, Rouen, France.*

Project  $O_1, O_2, P$  orthogonally onto  $t_1$  at  $U_1, U_2, P_1$ , respectively. Since  $P$  is the mid-point of  $O_1O_2$ , we see that  $P_1$  is the mid-point of  $U_1U_2$ ; thus,  $P_1U_1^2 = P_1U_2^2$ . Since  $U_1$  and  $U_2$  are the points where  $t_1$  is tangent to  $\Gamma_1$  and  $\Gamma_2$ , it follows that  $P_1$  has the same power with respect to  $\Gamma_1$  and  $\Gamma_2$ . Therefore,  $P_1$  lies on the radical axis  $\ell$  of  $\Gamma_1$  and  $\Gamma_2$ . Similarly, using the projections  $P_2$  and  $P_c$  of  $P$  onto  $t_2$  and  $t_c$ , we see that  $P_1, P_2, P_c$  are all on  $\ell$ . Since the projections of  $P$  on the sides of the triangle  $E_1QE_2$  are collinear,  $P$  is on the circumcircle of  $\triangle E_1QE_2$  (and  $\ell$  is the Simson line of  $P$ ). The result follows.

*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.*

*Several readers pointed out that since  $O_1$  and  $O_2$  are the incentre and excentre of  $\triangle E_1QE_2$ , Kotchetov's problem is a known theorem: The circumcircle of a triangle contains the mid-point of each segment joining the incentre to an excentre. See, for example, Nathan Altshiller Court, College Geometry, Theorem 122, page 76, or Roger A. Johnson, Advanced Euclidean Geometry, Section 292, page 185.*

**2967.** [2004 : 368, 371] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let  $a_1, a_2, \dots, a_n$  be positive real numbers, and let

$$E_n = \sum_{i=1}^n \left( \sum_{j=0}^{n-1} a_i^j \right)^{-1}.$$

If  $r = \sqrt[n]{a_1 a_2 \cdots a_n} \geq 1$ , prove that  $E_n \geq n \left( \sum_{j=0}^{n-1} r^j \right)^{-1}$  for:

(a)  $n = 2$ ,                      (b)  $n = 3$ ,                      (c)★  $n \geq 4$ .

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

(a) Let  $x = \frac{\sqrt{a_2}}{\sqrt{a_2} + \sqrt{a_1}}$  and  $y = \frac{\sqrt{a_1}}{\sqrt{a_2} + \sqrt{a_1}}$ . Then  $x + y = 1$ , and  $E_2 = \frac{x}{x + ry} + \frac{y}{y + rx}$ . Applying Jensen's Inequality to the convex function

$1/t$  on  $(0, \infty)$ , we see that

$$E_2 \geq \frac{1}{x(x+ry) + y(y+rx)} = \frac{1}{(x+y)^2 + 2(r-1)xy}.$$

By the AM–GM Inequality, we have  $xy \leq \left(\frac{x+y}{2}\right)^2 = \frac{1}{4}$ . Hence,

$$E_2 \geq \frac{1}{1 + \frac{1}{2}(r-1)} = \frac{2}{1+r}.$$

(b) Let  $k = \sum_{i=1}^3 \sqrt[3]{a_i^2 a_{i+1}^2}$ , and let  $x_i = \frac{1}{k} \sqrt[3]{a_{i+1}^2 a_{i+2}^2}$  for  $i = 1, 2, 3$ , where all subscripts are taken modulo 3. Then  $x_1 + x_2 + x_3 = 1$ , and

$$E_3 = \sum_{i=1}^3 \frac{x_i}{x_i + r\sqrt{x_{i+1}x_{i+2}} + r^2 \left(\frac{x_{i+1}x_{i+2}}{x_i}\right)}.$$

Applying Jensen's Inequality to the convex function  $1/t$  on  $(0, \infty)$ , we see that

$$\begin{aligned} E_3 &\geq \frac{1}{\sum_{i=1}^3 x_i \left( x_i + r\sqrt{x_{i+1}x_{i+2}} + r^2 \left(\frac{x_{i+1}x_{i+2}}{x_i}\right) \right)} \\ &= \frac{1}{\left(\sum_{i=1}^3 x_i\right)^2 + r \sum_{i=1}^3 x_i \sqrt{x_{i+1}x_{i+2}} + (r^2 - 2) \sum_{i=1}^3 x_i x_{i+1}}. \end{aligned}$$

By the AM–GM Inequality, we have

$$\begin{aligned} \sum_{i=1}^3 x_i \sqrt{x_{i+1}x_{i+2}} &\leq \sum_{i=1}^3 x_i x_{i+1} \leq \frac{1}{3} \left( \sum_{i=1}^3 x_i^2 + 2 \sum_{i=1}^3 x_i x_{i+1} \right) \\ &= \frac{1}{3} (x_1 + x_2 + x_3)^2 = \frac{1}{3}. \end{aligned}$$

Hence,

$$\begin{aligned} E_3 &\geq \frac{1}{1 + (r^2 + r - 2) \left( \sum_{i=1}^3 x_i x_{i+1} \right)} \\ &\geq \frac{1}{1 + \frac{1}{3}(r^2 + r - 2)} = \frac{3}{1 + r + r^2}. \end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; and the proposer.

Part (c) remains open. Janous provided a partial solution where each  $a_i \geq 1$ .

**2968.** [2004 : 368, 371] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let  $a_1, a_2, \dots, a_n$  be positive real numbers, and let

$$E_n = \frac{1 + a_1 a_2}{1 + a_1} + \frac{1 + a_2 a_3}{1 + a_2} + \dots + \frac{1 + a_n a_1}{1 + a_n}.$$

Let  $r = \sqrt[n]{a_1 a_2 \dots a_n} \geq 1$ .

(a) Prove that  $E_n \geq \frac{n(1+r^2)}{1+r}$  for  $n = 3$  and  $n = 4$ .

(b)★ Prove or disprove that  $E_n \geq \frac{n(1+r^2)}{1+r}$  for  $n = 5$ .

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

(a) Case  $n = 3$ .

Let  $k = \sum_{i=1}^3 \sqrt[3]{a_i a_{i+1}^2}$ , and let  $x_i = \frac{1}{k} \sqrt[3]{a_{i+1} a_{i+2}^2}$  for  $i = 1, 2, 3$ , where all subscripts are taken modulo 3. Then  $x_1 + x_2 + x_3 = 1$  and

$$E_3 = \sum_{i=1}^3 \frac{x_i + r^2 x_{i+2}}{x_i + r x_{i+1}}.$$

Applying Jensen's Inequality to the convex function  $1/t$  on  $(0, \infty)$ , we see that

$$\begin{aligned} E_3 &\geq \frac{\left(\sum_{i=1}^3 (x_i + r^2 x_{i+2})\right)^2}{\sum_{i=1}^3 (x_i + r^2 x_{i+2})(x_i + r x_{i+1})} \\ &= \frac{(1+r^2)^2}{(x_1 + x_2 + x_3)^2 + (r^3 + r^2 + r - 2) \sum_{i=1}^3 x_i x_{i+1}}. \end{aligned}$$

Now

$$1 = (x_1 + x_2 + x_3)^2 = \sum_{i=1}^3 x_i^2 + 2 \sum_{i=1}^3 x_i x_{i+1} \geq 3 \sum_{i=1}^3 x_i x_{i+1};$$

thus,

$$E_3 \geq \frac{(1+r^2)^2}{1 + \frac{1}{3}(r^3 + r^2 + r - 2)} = \frac{3(1+r^2)}{1+r}.$$

Case  $n = 4$ .

Let  $k = \sum_{i=1}^4 \sqrt[4]{a_i a_{i+1}^2 a_{i+2}^3}$ , and let  $x_i = \frac{1}{k} \sqrt[4]{a_{i+1} a_{i+2}^2 a_{i+3}^3}$  for  $i = 1, 2, 3, 4$ , where all subscripts are taken modulo 4. Then  $x_1 + x_2 + x_3 + x_4 = 1$ ,



and

$$E_4 = \sum_{i=1}^4 \frac{x_i + r^2 x_{i+2}}{x_i + r x_{i+1}}.$$

Applying Jensen's Inequality to the convex function  $1/t$  on  $(0, \infty)$ , we see that

$$\begin{aligned} E_4 &\geq \frac{\left(\sum_{i=1}^4 (x_i + r^2 x_{i+2})\right)^2}{\sum_{i=1}^4 (x_i + r^2 x_{i+2})(x_i + r x_{i+1})} \\ &= \frac{(1+r^2)^2}{(x_1^2 + x_2^2 + x_3^2 + x_4^2) + (r^3 + r) \left(\sum_{i=1}^4 x_i x_{i+1}\right) + r^2(2x_1 x_3 + 2x_2 x_4)}. \end{aligned}$$

Now

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= (x_1 + x_2 + x_3 + x_4)^2 - (x_1 + x_2)(x_3 + x_4) \\ &\quad - (x_1 + x_3)(x_2 + x_4) - (x_1 + x_4)(x_2 + x_3), \\ \sum_{i=1}^4 x_i x_{i+1} &= (x_1 + x_3)(x_2 + x_4), \\ 2x_1 x_3 + 2x_2 x_4 &= (x_1 + x_2)(x_3 + x_4) + (x_1 + x_4)(x_2 + x_3) \\ &\quad - (x_1 + x_3)(x_2 + x_4). \end{aligned}$$

Hence,

$$E_4 \geq \frac{(1+r^2)^2}{(x_1 + x_2 + x_3 + x_4)^2 + (r^2 + 1)(r-1)(x_1 + x_3)(x_2 + x_4) + (r^2 - 1)[(x_1 + x_2)(x_3 + x_4) + (x_1 + x_4)(x_2 + x_3)]}.$$

Finally, by the AM-GM Inequality,

$$(x_1 + x_3)(x_2 + x_4) \leq \frac{1}{4}[(x_1 + x_3) + (x_2 + x_4)]^2 \leq \frac{1}{4}.$$

Similarly, we have  $(x_1 + x_2)(x_3 + x_4) \leq \frac{1}{4}$  and  $(x_1 + x_4)(x_2 + x_3) \leq \frac{1}{4}$ . Thus,

$$E_4 \geq \frac{(1+r^2)^2}{1 + \frac{1}{4}(r^3 - r^2 + r - 1) + \frac{1}{2}(r^2 - 1)} = \frac{4(1+r^2)}{1+r}.$$

Also solved by MIHÁLY BENCZE, Brasov, Romania; and the proposer.  
Part (b) remains open.

**2969.** [2004 : 368, 371] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let  $a, b, c, d$ , and  $r$  be positive real numbers such that  $r = \sqrt[4]{abcd} \geq 1$ . Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq \frac{4}{(1+r)^2}.$$

*Solution by Arkady Alt, San Jose, CA, USA.*

I suggest the following generalization. For any natural  $n \geq 2$ , let  $a_1, a_2, \dots, a_n > 0$  such that  $a_1 a_2 \cdots a_n = r^n$ . Then

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \geq \frac{n}{(1 + \sqrt[n]{a_1 a_2 \cdots a_n})^2}$$

if and only if  $r \geq \sqrt{n} - 1$ .

[*Ed.*: In fact, the condition is not sufficient when  $n = 2$ . It is possible to find  $\varepsilon > 0$  such that  $a_1 = \sqrt{2} - 1$ ,  $a_2 = a_1 + \varepsilon$ , and  $r > \sqrt{2} - 1$ , but the inequality fails. The slightly stronger condition  $r \geq 0.5$  is sufficient when  $n = 2$ . Moreover, the inductive step still holds for  $n = 2$  using this stronger condition. That is, for  $n > 2$ ,  $r \geq \sqrt{n} - 1$  is sufficient for the inequality to hold. The editor has not determined the minimum sufficient value of  $r$  in the case  $n = 2$ .]

We begin with necessity. From the supposition that the inequality holds for all  $a_1, a_2, \dots, a_n > 0$  with  $a_1 a_2 \cdots a_n = r^n$ , and by setting  $a_1 = a_2 = \cdots = a_{n-1} = m$ ,  $a_n = \frac{r^n}{m^{n-1}}$ , for  $m \in \mathbb{R}^+$ , we obtain

$$\frac{n-1}{(1+m)^2} + \frac{m^{2(n-1)}}{(m^{n-1} + r^n)^2} \geq \frac{n}{(1+r)^2},$$

which holds for all positive  $m$ . Thus,

$$\lim_{m \rightarrow \infty} \left( \frac{n-1}{(1+m)^2} + \frac{m^{2(n-1)}}{(m^{n-1} + r^n)^2} \right) = 1 \geq \frac{n}{(1+r)^2},$$

which implies  $r \geq \sqrt{n} - 1$ .

We prove sufficiency by mathematical induction on  $n \geq 2$ .

Let  $n = 2$  and  $a, b > 0$  such that  $ab = r^2$  with  $r \geq 0.5$ .

Set  $x = a + b$ . Then  $x \geq 2r$ . Since

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} = \frac{2 + 2(a+b) + (a+b)^2 - 2ab}{(1+a+b+ab)^2},$$

the inequality can be rewritten in the form:

$$\frac{2 + 2x + x^2 - 2r^2}{(1 + r^2 + x)^2} \geq \frac{2}{(1+r)^2}$$

$$(1+r)^2 (2 + 2x + x^2 - 2r^2) \geq 2(1+x+r^2)^2.$$

This inequality holds if and only if

$$\begin{aligned}
0 &\leq (1+r)^2(2+2x+x^2-2r^2) - 2(1+x+r^2)^2 \\
&= x^2((1+r)^2-2) - 2x(2(1+r^2)-(1+r)^2) \\
&\quad + (2-2r^2)(1+r)^2 - 2(1+r^2)^2 \\
&= x^2(r^2+2r-1) - 2x(r^2-2r+1) - 4r^4 - 4r^3 - 4r^2 + 4r \\
&= (x-2r)(x(r^2+2r-1) + 2(r^3+r^2+r-1)).
\end{aligned}$$

Since  $r^2+2r-1 \geq 0$  (this follows from  $r \geq \sqrt{2}-1$ ) and  $x \geq 2r$ , we have

$$\begin{aligned}
x(r^2+2r-1) + 2(r^3+r^2+r-1) &\geq 2r(r^2+2r-1) + 2r^3+2r^2+2r-2 \\
&= 2(2r^3+3r^2-1) = 2(r+1)^2(2r-1) \geq 0.
\end{aligned}$$

Thus,

$$(x-2r)(x(r^2+2r-1) + 2(r^3+r^2+r-1)) \geq 0.$$

Let  $a_1, a_2, \dots, a_n, a_{n+1} > 0$  and  $a_1 a_2 \cdots a_{n+1} = r^{n+1}$ , where  $r \geq \sqrt{n+1}-1$ . Due to symmetry of the inequality, we can suppose that  $a_1 \geq a_2 \geq \dots \geq a_n \geq a_{n+1} > 0$ .

Set  $x = \sqrt[n]{a_1 a_2 \cdots a_n}$ ; then  $a_{n+1} = \frac{r^{n+1}}{x^n}$ . Since

$$x \geq a_{n+1} \iff x^{n+1} \geq r^{n+1} \iff x \geq r,$$

we have  $x \geq r$ .

Given  $x \geq \sqrt{n+1}-1 > \sqrt{n}-1$  and the induction hypothesis, we obtain the inequality:

$$\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} \geq \frac{n}{(1+x)^2}.$$

[Ed.: Note that, for  $n=2$ , we have  $x \geq \sqrt{3}-1 > 0.5$ ; hence, the inequality does indeed hold. For  $n > 2$ ,  $x > \sqrt{n}-1$ .]

Therefore,

$$\begin{aligned}
\frac{1}{(1+a_1)^2} + \frac{1}{(1+a_2)^2} + \cdots + \frac{1}{(1+a_n)^2} + \frac{1}{(1+a_{n+1})^2} \\
\geq \frac{n}{(1+x)^2} + \frac{x^{2n}}{(x^n+r^{n+1})^2},
\end{aligned}$$

and it is enough to prove that, for all  $x \geq r \geq \sqrt{n+1}-1$ ,

$$\frac{n}{(1+x)^2} + \frac{x^{2n}}{(x^n+r^{n+1})^2} \geq \frac{n+1}{(1+r)^2}.$$

Let  $h(x) = \frac{n}{(1+x)^2} + \frac{x^{2n}}{(x^n + r^{n+1})^2}$ . Then

$$h'(x) = \frac{2n(x^{n+1} - r^{n+1})(x^{n+1}r^{n+1} + 3x^n r^{n+1} + r^{2n+2} - x^{2n-1})}{(1+x)^3(x^n + r^{n+1})^3}.$$

Now everything depends on the behaviour of the polynomial

$$P_n(x) = x^{n+1}r^{n+1} + 3x^n r^{n+1} + r^{2n+2} - x^{2n-1}.$$

Note that

$$\begin{aligned} x^{n+1}r^{n+1} + 3x^n r^{n+1} + r^{2n+2} - x^{2n-1} &= 0 \\ \text{or } r^{n+1} + \frac{3r^{n+1}}{x} + \frac{r^{2n+2}}{x^{n+1}} - x^{n-2} &= 0. \end{aligned}$$

Set  $\phi(x) = r^{n+1} + \frac{3r^{n+1}}{x} + \frac{r^{2n+2}}{x^{n+1}} - x^{n-2}$ .

Since  $r \geq \sqrt{n+1} - 1 > \frac{1}{2}$  for  $n \geq 2$ , we have

$$\begin{aligned} P_n(r) &= 2r^{2n+2} + 3r^{2n+1} - r^{2n-1} \\ &= r^{2n-1}(2r^3 + 3r^2 - 1) \\ &= r^{2n-1}(r+1)^2(2r-1) > 0 \\ \iff \phi(r) &> 0. \end{aligned}$$

Since  $\phi(x)$  is continuous on  $(0, \infty)$ ,  $\phi(x)$  strictly decreases on  $[r, \infty)$ , and  $\phi(\infty)\phi(r) < 0$ , there is only one point,  $x_0$ , in  $(r, \infty)$  such that  $\phi(x_0) = 0$ , or equivalently  $P_n(x_0) = 0$ .

Moreover,  $\phi(x) > \phi(x_0) = 0$  is equivalent to  $P_n(x) > 0$  for all  $x \in [r, x_0)$ , and  $0 = \phi(x_0) > \phi(x)$  is equivalent to  $P_n(x) < 0$  for all  $x \in (x_0, \infty)$ .

Since

$$\min_{x \in [r, x_0]} h(x) = h(r) = \frac{n}{(1+r)^2} + \frac{r^{2n}}{(r^n + r^{n+1})^2} = \frac{n+1}{(1+r)^2},$$

and, for any  $x \in [x_0, \infty)$ ,

$$h(x) > \lim_{x \rightarrow \infty} h(x) = 1 \geq \frac{n+1}{(1+r)^2} = h(r),$$

we obtain

$$\min_{x \in [r, \infty)} h(x) = h(r) = \frac{n+1}{(1+r)^2}.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIHÁLY BENCZE, Brasov, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

**2970.** [2004 : 368, 371] *Proposed by Titu Zvonaru, Comănești, Romania.*

If  $m$  and  $n$  are positive integers such that  $m \geq n$ , and if  $a, b, c > 0$ , prove that

$$\frac{a^m}{b^m + c^m} + \frac{b^m}{c^m + a^m} + \frac{c^m}{a^m + b^m} \geq \frac{a^n}{b^n + c^n} + \frac{b^n}{c^n + a^n} + \frac{c^n}{a^n + b^n}.$$

**I. Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA, modified slightly by the editor.**

Without loss of generality, we may assume that  $a \geq b \geq c$ . Let  $x = a^n$ ,  $y = b^n$ ,  $z = c^n$ , and set  $r = \frac{m}{n}$ . Then  $x \geq y \geq z$  and  $r \geq 1$ . Let

$$F = \sum_{\text{cyclic}} \frac{a^m}{b^m + c^m} - \sum_{\text{cyclic}} \frac{a^n}{b^n + c^n}. \text{ Then } F = \sum_{\text{cyclic}} \frac{x^r}{y^r + z^r} - \sum_{\text{cyclic}} \frac{x}{y + z}.$$

We have

$$\frac{x^r}{y^r + z^r} - \frac{x}{y + z} = \frac{xy(x^{r-1} - y^{r-1})}{(y + z)(y^r + z^r)} + \frac{xz(x^{r-1} - z^{r-1})}{(y + z)(y^r + z^r)}. \quad (1)$$

Similarly,

$$\frac{y^r}{z^r + x^r} - \frac{y}{z + x} = \frac{yz(y^{r-1} - z^{r-1})}{(z + x)(z^r + x^r)} + \frac{yx(y^{r-1} - x^{r-1})}{(z + x)(z^r + x^r)} \quad (2)$$

and

$$\frac{z^r}{x^r + y^r} - \frac{z}{x + y} = \frac{zx(z^{r-1} - x^{r-1})}{(x + y)(x^r + y^r)} + \frac{zy(z^{r-1} - y^{r-1})}{(x + y)(x^r + y^r)}. \quad (3)$$

Adding (1), (2), and (3), we obtain

$$\begin{aligned} F &= xy(x^{r-1} - y^{r-1}) \left( \frac{1}{(y + z)(y^r + z^r)} - \frac{1}{(z + x)(z^r + x^r)} \right) \\ &\quad + yz(y^{r-1} - z^{r-1}) \left( \frac{1}{(z + x)(z^r + x^r)} - \frac{1}{(x + y)(x^r + y^r)} \right) \\ &\quad + xz(x^{r-1} - z^{r-1}) \left( \frac{1}{(y + z)(y^r + z^r)} - \frac{1}{(x + y)(x^r + y^r)} \right). \end{aligned}$$

Since  $x \geq y \geq z$  and  $r \geq 1$ , we see that all the terms on the right side of the last expression above are non-negative. Hence,  $F \geq 0$ .

**II. Composite of similar solutions by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and Li Zhou, Polk Community College, Winter Haven, FL, USA.**

We show that the result actually holds for all reals  $m$  and  $n$  with  $m \geq n \geq 0$ . Due to symmetry, we may assume that  $a \geq b \geq c > 0$ .

Let  $x_m = a^m/(a^m + b^m + c^m)$ ,  $y_m = b^m/(a^m + b^m + c^m)$ , and  $z_m = c^m/(a^m + b^m + c^m)$ . Then  $x_m \geq y_m \geq z_m$  and  $x_n \geq y_n \geq z_n$ .

Furthermore,

$$\begin{aligned} x_m - x_n &= \frac{a^n b^n (a^{m-n} - b^{m-n}) + a^n c^n (a^{m-n} - c^{m-n})}{(a^m + b^m + c^m)(a^n + b^n + c^n)} \geq 0 \\ (x_m + y_m) - (x_n + y_n) &= \frac{a^n c^n (a^{m-n} - c^{m-n}) + b^n c^n (b^{m-n} - c^{m-n})}{(a^m + b^m + c^m)(a^n + b^n + c^n)} \geq 0, \end{aligned}$$

and  $(x_m + y_m + z_m) - (x_n + y_n + z_n) = 1 - 1 = 0$ . Thus,  $(x_m, y_m, z_m)$  majorizes  $(x_n, y_n, z_n)$ .

Consider the function  $f(t) = t/(1-t)$ . It is easy to check that  $f$  is continuous and convex on  $(0, 1)$ . By the Majorization Inequality, we have

$$f(x_m) + f(y_m) + f(z_m) \geq f(x_n) + f(y_n) + f(z_n),$$

which is equivalent to the given inequality.

[Ed: Zhao uses the term Karamata's Majorization Inequality. However, this inequality is usually attributed to Hardy, Pólya, and Littlewood. See, for example, Proposition B1 on page 108 of the book *Inequalities: Theory of Majorization and its Applications* by Albert W. Marshall and Ingram Olkin, Academic Press, 1979.]

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; VASILE CÎRTOAJE, University of Ploiesti, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incomplete solution.

**2971.** [2004 : 369, 371] Proposed by Michel Bataille, Rouen, France.

For  $a, b, c \in (0, 1)$ , find the least upper bound and the greatest lower bound of  $a + b + c + abc$ , subject to the constraint  $ab + bc + ca = 1$ .

*Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Note first that

$$\begin{aligned} a + b + c + abc &= 1 + ab + bc + ca - (1-a)(1-b)(1-c) \\ &= 2 - (1-a)(1-b)(1-c) \leq 2. \end{aligned}$$

If we let  $a = b = t$  and  $c = \frac{1-t^2}{2t}$ , for  $t \in (0, 1)$ , then

$$ab + bc + ca = (a+b)c + ab = 1 - t^2 + t^2 = 1.$$

Since  $(1-a)(1-b)(1-c) \rightarrow 0^+$  as  $t \rightarrow 1^-$ , we conclude that the required least upper bound is 2, which is not attainable.

On the other hand,

$$(a + b + c)^2 = \frac{1}{2}((a - b)^2 + (b - c)^2 + (c - a)^2 + 6(ab + bc + ca)) \geq 3.$$

Thus, by the AM–GM Inequality, we have

$$(1 - a)(1 - b)(1 - c) \leq \left(1 - \frac{a + b + c}{3}\right)^3 \leq \left(1 - \frac{\sqrt{3}}{3}\right)^3 = 2 - \frac{10\sqrt{3}}{9}.$$

Hence, the required greatest lower bound is  $\frac{10\sqrt{3}}{9}$ , which is attained when  $a = b = c = \sqrt{3}/3$ .

*Also solved by ANGELO STATE UNIVERSITY PROBLEM GROUP, San Angelo, TX, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VINAYAK MURALIDHAR, student, Corona del Sol High School, Tempe, AZ, USA; PANOS E. TSAOUSSOGLU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposers. There were two incomplete solutions and one partly incorrect solution.*

*The Angelo State University Problem Group showed that the following result is an immediate consequence of the answers to the current problem: if  $\triangle ABC$  is an acute-angled triangle, then*

$$\frac{10\sqrt{3}}{9} \leq \tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right) + \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) < 2,$$

*with equality in the left inequality if and only if the triangle is equilateral.*

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