THE OLYMPIAD CORNER

No. 248

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We begin this number of the *Corner* with the problems of the 2002 Yugoslav Mathematical Olympiad. Thanks go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining them for our use.

2002 YUGOSLAV MATHEMATICAL OLYMPIAD

1. Let a, b, and c be positive numbers, and let n and k be positive integers. Prove the inequality:

$$\frac{a^{n+k}}{b^n}+\frac{b^{n+k}}{c^n}+\frac{c^{n+k}}{a^n} \;\geq\; a^k+b^k+c^k\,.$$

2. Let (f_n) be a sequence defined by: $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$, for $n \geq 1$. Prove that the area of the triangle with side lengths $\sqrt{f_{2n+1}}$, $\sqrt{f_{2n+2}}$, and $\sqrt{f_{2n+3}}$ equals $\frac{1}{2}$.

3. Let $ABCD$ be a rhombus with $\angle BAD = 60^\circ$. Points S and R lie inside triangles ABD and DBC, respectively, such that $\angle SBR = \angle RDS = 60^\circ$. Prove that $SR^2 \geq AS \cdot CR$.

4. Does there exist a positive integer k such that the digits 3, 4, 5, and 6 do not appear in the decimal representation of the number $2002! \cdot k$?

Next we give the Yugoslav Qualification for IMO 2002, First and Second Rounds. Thanks again go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining them for us.

YUGOSLAV QUALIFICATION FOR IMO 2002

First Round

1. A man standing at the point $(1,1)$ in the coordinate plane wants to find an object that lies at some point (α,β), where $\alpha \in \{1,\,2,\,\ldots,\,m\}$, and $\beta \in \{1,\,2,\,\ldots,\,n\}$. After finding the object, he will return to the starting point. Find the minimal worst case time needed for doing this job, if he does not know exactly at which point the object lies, and if he can move in any direction with velocity not greater than one.

2. Let p be the semiperimeter of the triangle ABC. Let the points E and F lie on the line AB such that $CE = CF = p$. Prove that the circumcircle of the triangle EFC and the circle that touches the side AB and the extension of the sides AC and BC of the triangle ABC meet in one point.

3. Let $\{x_n\}_{n\geq 2}$, be a sequence such that $x_2 = 1$, $x_3 = 1$, and, for $n \geq 3$,

$$(n+1)(n-2)x_{n+1} = n(n^2-n-1)x_n - (n-1)^3 x_{n-1}\,.$$

Prove that x_n is an integer if and only if n is a prime.

Second Round

1. What is the maximal value of the expression $a+b+c+abc$, if a, b, c are non-negative numbers such that $a^2+b^2+c^2+abc \leq 4$?

2. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle ABC = \angle BCD$. Let O and H be the circumcentre and the orthocentre, respectively, of triangle ABC. Prove that the points O, H, and D are collinear.

3. For any positive integer n, let $f(n)$ denote the number of distinct possible choices for plus and minus signs such that $\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = 0$ holds true. Prove that:

(a) $f(n) = 0$, for $n \equiv 1, 2 \pmod 4$;

(b) $f(n) \geq 2^{\frac{n}{2}-1}$, for $n \equiv 0, 3 \pmod 4$.

To round out the contests presented, we give the problems of the Midi Finale 2002 and the Maxi Finale 2002 of the 27$^{\text{ième}}$ Olympiade Mathématique Belge. Thanks again go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining these problems.

VINGT-SEPTIÈME OLYMPIADE MATHÉMATIQUE BELGE

Midi Finale

Mercredi 24 avril 2002

1. Soit M le milieu de la base $[AB]$ d'un trapèze isocèle $ABCD$ et E le point d'intersection de MD et de AC. Si M est le centre du cercle circonscrit au trapèze et si $[AD]$ et $[DE]$ ont la même longueur, déterminer l'amplitude de l'angle $\angle DAB$.

2. La somme de quatre nombres réels est nulle ; la somme de leurs cubes est également nulle. Est-il vrai qu'alors deux de ces quatre nombres sont nécessairement opposés ?

3. (a) Existe-t-il quatre nombres naturels distincts non nuls tels que la somme de trois quelconques d'entre eux soit toujours un nombre premier ?

(b) Existe-t-il cinq nombres naturels distincts non nuls tels que la somme de trois quelconques d'entre eux soit toujours un nombre premier ?

4. Soit un rectangle $ABCD$, P un point situé sur un des côtés de ce rectangle, E et F les pieds des hauteurs abaissées de P sur les diagonales du rectangle. Démontrer que la somme $|PE| + |PF|$ reste constante lorsque P parcourt le périmètre de $ABCD$.

Maxi Finale

Mercredi 24 avril 2002

1. Soit la suite $(a_n)_{n \in N}$ telle que $a_n = n + \lfloor \sqrt{n} \rfloor$ pour tout $n \in \mathbb{N}$. Déterminer le plus petit entier naturel k pour lequel $a_k, a_{k+1}, \ldots, a_{k+2001}$ constituent une suite de 2002 entiers consécutifs. (Note : $\lfloor x \rfloor$ désigne le plus grand entier plus petit ou égal à x.)

2. (a) Dans le plan, soient $AB_1C_1D_1$ et $AB_2C_2D_2$ deux carrés ayant un sommet commun (les sommets sont cités dans le même sens). Si B, C et D sont respectivement les milieux des segments $[B_1B_2]$, $[C_1C_2]$ et $[D_1D_2]$, le quadrilatère $ABCD$ est-il aussi un carré ?

(b) Qu'en est-il si les sommets des carrés $AB_1C_1D_1$ et $AB_2C_2D_2$ sont cités en sens opposés ?

3. Voici une vue partielle d'une table de multiplication dans laquelle un tableau rectangulaire a été sélectionné.

$$\begin{array}{|ccccc|cc}
\hline
1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
2 & 4 & 6 & 8 & 10 & 12 & \cdots \\
3 & 6 & 9 & 12 & 15 & 18 & \cdots \\
4 & 8 & 12 & 16 & 20 & 24 & \cdots \\
\hline
\multicolumn{1}{c}{5} & 10 & 15 & 20 & \multicolumn{1}{c}{25} & 30 & \cdots \\
\multicolumn{1}{c}{\vdots} & \vdots & \vdots & \vdots & \multicolumn{1}{c}{\vdots} & \vdots & \ddots
\end{array}$$

Pour chaque tableau dont l'élément du coin supérieur gauche et celui du coin inférieur droit sont respectivement 1 et 2002, on calcule la somme de tous ses éléments. Quelle est la plus petite des sommes ainsi obtenues ?

4. Trouver tous les nombres premiers a et b tels que $a^{a+1} + b^{b+1}$ est aussi un nombre premier.

Next we turn to solutions by our readers to the problems of the 32nd Austrian Mathematics Olympiad given in the October 2003 number of the *Corner* [2003 : 374–375].

1. Prove that

$$\frac{1}{25}\sum_{k=0}^{2001}\left\lfloor\frac{2^k}{25}\right\rfloor$$

is a natural number, where $\lfloor x\rfloor$ denotes the greatest whole number less than or equal to x.

Solved by Christopher J. Bradley, Bristol, UK; Pierre Bornsztein, Maisons-Laffitte, France; Mike Spivey, Samford University, Birmingham, AL, USA; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Spivey's write-up.

Since $\lfloor 2^k/25\rfloor$ is the quotient when 2^k is divided by 25, we have, by the Division Algorithm, $2^k = 25\lfloor 2^k/25\rfloor + r_k$, where r_k is the unique integer such that $r_k \equiv 2^k \pmod{25}$ and $0 \le r_k < 25$. Therefore,

$$25\sum_{k=0}^{2001}\left\lfloor\frac{2^k}{25}\right\rfloor = \sum_{k=0}^{2001}\left(2^k - r_k\right) = 2^{2002} - 1 - \sum_{k=0}^{2001} r_k\,. \tag{1}$$

If n is a natural number such that 2 and n are relatively prime, then, by Euler's Theorem, $2^{\phi(n)} \equiv 1 \pmod{n}$, where ϕ is Euler's totient function. In particular, taking $n = 25$ and noting that $\phi(25) = \phi\left(5^2\right) = 5^2 - 5 = 20$, we find that $2^{20} \equiv 1 \pmod{25}$. Then $2^{2000} = \left(2^{20}\right)^{100} \equiv 1 \pmod{25}$ and $2^{2001} = 2\cdot 2^{2000} \equiv 2 \pmod{25}$. Thus, $r_{2000} = 1$ and $r_{2001} = 2$.

As k runs through any 20 consecutive natural numbers, r_k runs through the 20 natural numbers that are less than 25 and relatively prime to 25. The sum of these 20 values for r_k is $\sum\limits_{i=1}^{25} i - (5 + 10 + 15 + 20 + 25) = 250$. Therefore,

$$\begin{aligned}\sum_{k=0}^{2001} r_k &= r_{2000} + r_{2001} + \sum_{k=0}^{1999} r_k = 1 + 2 + \frac{2000}{20}(250)\\ &\equiv 1 + 2 + 25000 \equiv 3 \pmod{625}\,.\end{aligned}$$

Applying Euler's Theorem again, we see that $2^{\phi(625)} \equiv 1 \pmod{625}$. Since $\phi(625) = \phi\left(5^4\right) = 5^4 - 5^3 = 500$, we have $2^{500} \equiv 1 \pmod{625}$. Then $2^{2000} = \left(2^{500}\right)^4 \equiv 1 \pmod{625}$, and hence $2^{2002} \equiv 4 \pmod{625}$.

Using our results in (1), we obtain

$$25\sum_{k=0}^{2001}\left\lfloor\frac{2^k}{25}\right\rfloor \equiv 4 - 1 - 3 = 0 \pmod{625}\,.$$

Then $\sum\limits_{k=0}^{2001}\left\lfloor\frac{2^k}{25}\right\rfloor$ is divisible by 25, which gives the desired result.

2. Determine all triplets of positive real numbers x, y, and z solving the system of equations

$$\begin{aligned} x+y+z &= 6\,, \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= 2-\frac{4}{xyz}\,. \end{aligned}$$

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; Skotidas Sotirios, Karditso, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's write-up.

Suppose that x, y, z satisfy the first equation, $x+y+z=6$. Then, applying the AM–HM Inequality to $1/x$, $1/y$, $1/z$, we have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;\geq\; \frac{9}{x+y+z} \;=\; \frac{3}{2}\,.$$

By applying the GM–HM Inequality to the same set of variables, we get

$$\frac{1}{xyz} \;\geq\; \left(\frac{3}{x+y+z}\right)^3 \;=\; \frac{1}{8}\,.$$

It follows that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;\geq\; \frac{3}{2} \;=\; 2-4\left(\frac{1}{8}\right) \;\geq\; 2-\frac{4}{xyz}\,,$$

with equality if and only if $x=y=z=2$. Comparing the above inequality with the second equation in the system, we see that $(2,2,2)$ is the only solution to the system.

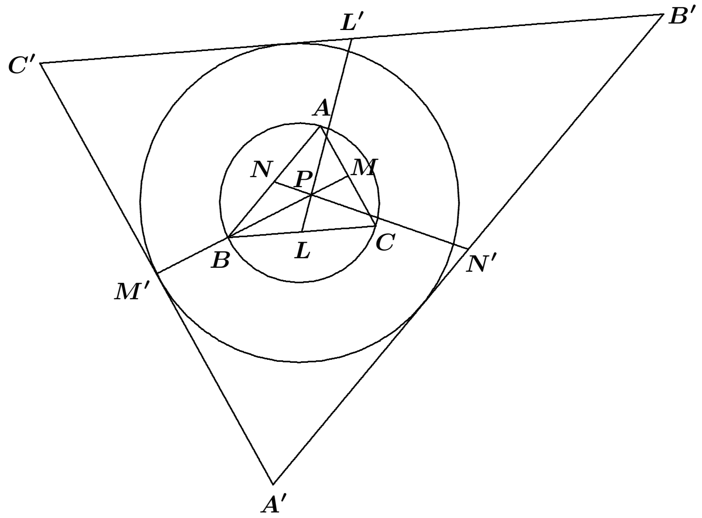
3. We are given a triangle ABC having $k(U,r)$ as its circumcircle. Next we construct the 'doubled' circle $k(U,2r)$ and its two tangents parallel to $c=AB$. Among them we select the one (and designate it c') for which C lies between c and c'. In a similar way we get the tangents a' and b'.

Let $A'B'C'$ be the triangle having its sides on a', b', and c', respectively. Prove: The lines joining the mid-points of corresponding sides of the two triangles intersect in a single point.

Solved by Christopher J. Bradley, Bristol, UK; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Bradley's write-up.

Triangles ABC and $A'B'C'$ have equal corresponding angles and are therefore similar. Corresponding sides of these triangles are parallel, which means that the triangles are homothetic. It follows that AA', BB', CC' are concurrent at a point P. Likewise, if L, M, N are the mid-points of ABC

and L', M', N' are the mid-points of $A'B'C'$, then (L, L'), (M, M'), and (N, N') are pairs of corresponding points; that is, LL', MM', NN' pass through P.



4. Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$, such that for all real numbers x and y the functional equation $f\big(f(x)^2 + f(y)\big) = x \cdot f(x) + y$ is satisfied.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's version.

Suppose f is a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$f\big(f(x)^2 + f(y)\big) \;=\; x \cdot f(x) + y\,. \tag{1}$$

By setting $x = 0$ in (1), we get

$$f\big(f(0)^2 + f(y)\big) \;=\; y\,.$$

Thus, f can take any value in $\mathbb{R}$; hence, it is surjective. Also, if $f(a) = f(b)$, then $a = f\big(f(0)^2 + f(a)\big) = f\big(f(0)^2 + f(b)\big) = b$. Therefore, f is injective. Thus, we see that f must be a bijective function.

Let h be a number such that $f(h) = 0$. By setting $x = y = h$ in (1), we get $f(0) = h$. Then, putting $x = y = 0$ in (1) gives $f(h^2 + h) = 0$. Since f is injective and $f(h^2 + h) = f(h)$, we must have $h^2 + h = h$; hence, $h = 0$. Thus, $f(0) = 0$.

By setting $x = 0$ in (1), we obtain

$$f\big(f(y)\big) \;=\; y\,. \tag{2}$$

And, by setting $y = 0$ in (1), we get

$$f\big(f(x)^2\big) \;=\; x \cdot f(x)\,. \tag{3}$$

Replacing x by $f(x)$ in (3) gives $f\Big(f\big(f(x)\big)^2\Big) \;=\; f(x) f\big(f(x)\big)$. Using (2), we simplify this to $f\left(x^2\right) = x \cdot f(x)$. By comparing this with (3), we get $f\left(f(x)^2\right) = f\left(x^2\right)$. Hence (since f is injective), $f(x)^2 = x^2$ for all x.

Suppose that there exist a and b in $\mathbb{R} \setminus \{0\}$ such that $f(a) = a$ and $f(b) = -b$. Setting $x = a$ and $y = b$ in (1) gives $f\left(a^2 - b\right) = a^2 + b$, which does not obey $f(x)^2 = x^2$. Therefore, either $f(x) = x$ for all x, or $f(x) = -x$ for all x.

5. Determine all whole numbers m for which all solutions of the equation $3x^3 - 3x^2 + m = 0$ are rational numbers.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Bradley's solution.

If $m = 0$, the given equation reduces to $x^3 - x^2 = 0$. The solutions of this equation are 0 (repeated) and 1, both of which are rational. Conversely, we will show that if all solutions of the equation are rational, then $m = 0$.

Suppose that all solutions are rational. Then there is a rational solution of the form $x = a/b$, where a and b are integers, $b > 0$, and a and b are relatively prime. Substituting such a solution into the equation, we get

$$3a^3 - 3a^2b + b^3m \;=\; 0\,.$$

Since b divides the last two terms, it follows that $b \mid 3a^3$. Then, since a and b are relatively prime, we must have $b = 1$ or $b = 3$. If $b = 3$, then $3a^3 - 9a^2 + 27m = 0$, or $a^3 - 3a^2 + 9m = 0$; hence, $3 \mid a$. This is a contradiction, since a and b are relatively prime. Therefore, $b = 1$. We conclude that $m = 3a^2(1 - a)$.

Setting $m = 3a^2(1-a)$ in the given equation, the equation becomes $x^3 - x^2 - a^3 + a^2 = 0$, or

$$(x - a)\left(x^2 + ax + a^2 - x - a\right) \;=\; 0\,.$$

The discriminant of the quadratic factor is $(1-a)(3a+1)$. Recalling that a is an integer, we see that the discriminant is non-negative only when $a = 0$ or $a = 1$. In both of these cases, we have $m = 0$.

6. We are given a semicircle s with diameter AB. On s we choose any two points C and D such that $AC = CD$. The tangent at C intersects line BD in a point E. Line AE intersects s at point F.

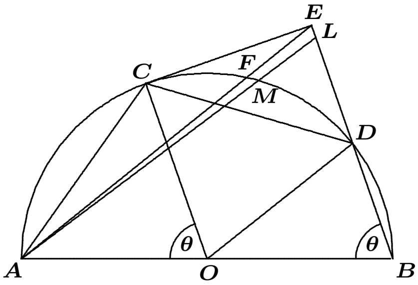
Prove that $CD < FD$.

Correction and solution by Christopher J. Bradley, Bristol, UK.

Presumably, the question should read: Prove that $CF < FD$ (and not $CD < FD$).

Take rectangular Cartesian coordinates with origin $O(0,0)$ at the centre of S and with $A(-1,0)$ and $B(1,0)$. Let θ be the angle subtended at O by AC. Then $0 < \theta < \frac{\pi}{2}$, and $C(-\cos\theta, \sin\theta)$ and $D(-\cos 2\theta, \sin 2\theta)$.

Letting M be the mid-point of the arc CD, we have $M\ \big(-\cos(\frac{3}{2}\theta), \sin(\frac{3}{2}\theta)\big)$.



Since $AC = CD$, the angle subtended at B by AD is equal to θ. Hence, BD is parallel to OC and has equation $x\sin\theta + y\cos\theta = \sin\theta$. The equation of the tangent at C is $-x\cos\theta + y\sin\theta = 1$. Letting E be the point at which these two lines meet, we find that the corrdinates of E are

$$x = 1 - (1+\cos\theta)\cos\theta \qquad \text{and} \qquad y = (1+\cos\theta)\sin\theta\,.$$

It follows that $\overrightarrow{BE} = \big[-(1+\cos\theta)\cos\theta, (1+\cos\theta)\sin\theta\big]$.

Now, $\overrightarrow{AM} = \big[1-\cos(\frac{3}{2}\theta), \sin(\frac{3}{2}\theta)\big]$. The slope of AM is therefore

$$\frac{\sin(\frac{3}{2}\theta)}{1-\cos(\frac{3}{2}\theta)} = \frac{2\sin(\frac{3}{4}\theta)\cos(\frac{3}{4}\theta)}{2\sin^2(\frac{3}{4}\theta)} = \cot(\tfrac{3}{4}\theta)\,,$$

and the equation of AM is $y\sin(\frac{3}{4}\theta) = (x+1)\cos(\frac{3}{4}\theta)$.

Let L be the point at which AM meets BD. The coordinates of L are $x = -\dfrac{\cos(\frac{7}{4}\theta)}{\cos(\frac{1}{4}\theta)}$ and

$$\begin{aligned} y &= \frac{2\cos(\frac{3}{4}\theta)\sin\theta}{\cos(\frac{1}{4}\theta)} = \frac{2\left(4\cos^3(\frac{1}{4}\theta) - 3\cos(\frac{1}{4}\theta)\right)\sin\theta}{\cos(\frac{1}{4}\theta)} \\ &= 2\left(4\cos^2(\tfrac{1}{4}\theta) - 3\right)\sin\theta\,. \end{aligned}$$

We aim to show that $BL < BE$, which implies that $CF < FD$ (see the diagram). It is thus sufficient to prove that

$$2\big(4\cos^2(\tfrac{1}{4}\theta) - 3\big)\sin\theta < (1+\cos\theta)\sin\theta\,.$$

Since $\theta \neq 0$, this is true if and only if

$$8\cos^2(\tfrac{1}{4}\theta) < 7 + \cos\theta\,. \tag{1}$$

Now consider $f(\theta) = 7 + \cos\theta - 8\cos^2(\frac{1}{4}\theta)$. We have $f(0) = 0$, and

$$\begin{aligned} \frac{df}{d\theta} &= -\sin\theta + 4\cos(\tfrac{1}{4}\theta)\sin(\tfrac{1}{4}\theta) = -\sin\theta + 2\sin(\tfrac{1}{2}\theta) \\ &= -2\sin(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta) + 2\sin(\tfrac{1}{2}\theta) = 2\sin(\tfrac{1}{2}\theta)\left(1-\cos(\tfrac{1}{2}\theta)\right)\,, \end{aligned}$$

which is positive for $0 < \theta \leq \frac{\pi}{2}$. Therefore, (1) holds, and the inequality is proved.

Next we turn to the November 2003 *Corner* and solutions to problems of the 14th Nordic Mathematical Contest given [2003 : 435].

1. In how many ways can the number 2000 be written as a sum of three positive, not necessarily different, integers? (Sums like $1+2+3$ and $3+1+2$, etc. are considered to be the same.)

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We use Bataille's solution.

We are required to find the number of triples (n_1, n_2, n_3) of integers such that $1 \le n_1 \le n_2 \le n_3$ and $n_1 + n_2 + n_3 = 2000$. This number is 333333, the integer nearest to $2000^2/12$, as we shall now prove.

More generally, let n be any integer with $n \ge 3$, and let t_n be the number of triples (n_1, n_2, n_3) with $1 \le n_1 \le n_2 \le n_3$ and $n_1+n_2+n_3 = n$. For $n \ge 2$, we similarly denote by d_n the number of pairs (n_1, n_2) such that $1 \le n_1 \le n_2$ and $n_1 + n_2 = n$.

Clearly, $d_2 = d_3 = 1$, and, more generally, it is readily seen that $d_n = n/2$ if n is even and $d_n = (n-1)/2$ if n is odd. Returning to t_n, we find that $t_3 = t_4 = 1$ and $t_5 = 2$. For $n \ge 6$, we have $t_n = d_{n-1} + t_{n-3}$, because there are d_{n-1} triples for which $n_1 = 1$, and the set of suitable triples with $n_1 \ge 2$ is obviously in bijection with the set of triples $(n_1 - 1, n_2 - 1, n_3 - 1)$ summing to $n - 3$.

Now, assume that $n = 6k + 2$ for some positive integer k. Then

$$\begin{aligned} t_n &= (t_{6k+2} - t_{6k-1}) + (t_{6k-1} - t_{6k-4}) + \cdots \\ &\qquad + (t_{11} - t_8) + (t_8 - t_5) + t_5 \\ &= d_{6k+1} + d_{6k-2} + d_{6k-5} + \cdots + d_7 + t_5 \\ &= 3k + (3k-1) + (3k-3) + (3k-4) + \cdots + 6 + 5 + 3 + 2 \\ &= \frac{k(3k+3)}{2} + \frac{k(3k+1)}{2} \;=\; k(3k+2) \;=\; \frac{n^2}{12} - \frac{1}{3}, \end{aligned}$$

which is the integer nearest to $n^2/12$. For $n = 2000$ this gives $t_n = 333333$. (Examining the cases $n = 6k$, $6k+1$, $6k+3$, $6k+4$, $6k+5$ in the same way, it is easy to show that t_n is always the integer nearest to $n^2/12$.)

2. The persons P_1, P_2, …, P_{n-1}, P_n sit around a table, in this order, and each one has a number of coins. At the start, P_1 has one coin more than P_2, P_2 has one coin more than P_3, etc., up to P_{n-1}, who has one coin more than P_n. Now P_1 gives one coin to P_2, who in turn gives two coins to P_3, etc., up to P_n, who gives n coins to P_1. The process continues in the same way: P_1 gives $n+1$ coins to P_2, P_2 gives $n+2$ coins to P_3, and so on. The transactions go on until someone has not enough coins to give away one coin more than he just received. At the moment when the process comes to an end in this manner, it turns out that there are two neighbours at the table one of whom has exactly five times as many coins as the other. Determine the number of persons and the number of coins circulating around the table.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

For $i = 1, 2, \ldots, n$ and $j \geq 0$, let $x_i(j)$ be the number of coins owned by P_i at the end of his j^{th} transaction. Since this transaction consists of P_i giving away one more coin than he has just received, we see that

$$x_i(j) \;=\; x_i(j-1) - 1\,. \qquad (1)$$

Let a be the number of coins that P_n has at the beginning. Then $x_i(0) = a + n - i$ for each i, and the total number of coins is

$$S \;=\; \sum_{i=1}^{n}(a+n-i) \;=\; an + \frac{n(n-1)}{2}\,.$$

It follows from equation (1) that the process will end when we have had $x_i(j) = 0$ for some i and j, and P_i is in the midst of his $(j+1)^{\text{st}}$ transaction (unable to give away one more coin than he has just received). In fact, since $x_1(0) > x_2(0) > \cdots > x_n(0) = a$, the process will end when P_n has to give to P_1 for the $(a+1)^{\text{th}}$ time. At this moment, using (1) again, we have $x_i(a+1) = n - i - 1$ for $i \leq n-1$, and P_n has $na + n - 1$ coins (having just received all of these from P_{n-1}).

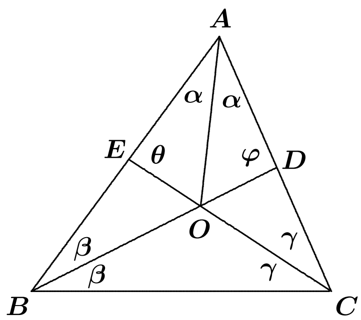
Since there are two neighbours at the table one of whom has exactly five times as many coins as the other, and since we may easily verify that $n - i \neq 5(n-i-1)$, we conclude that $na + n - 1 = 5x_1(a+1) = 5(n-2)$. This yields $9 = n(4-a)$. Thus, either $n = 3$ and $a = 1$, or $n = 9$ and $a = 3$. That is,

$$\begin{cases} n \;=\; 3 \\ S \;=\; 6 \end{cases} \quad \text{or} \quad \begin{cases} n \;=\; 9 \\ S \;=\; 63\,. \end{cases}$$

(Obviously, the argument is "if and only if".)

3. In the triangle ABC, the bisector of angle B meets AC at D, and the bisector of angle C meets AB at E. The bisectors intersect at O, and $OD = OE$. Prove that either $\triangle ABC$ is isosceles or $\angle BAC = 60^\circ$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Christopher J. Bradley, Bristol, UK; and Geoffrey A. Kandall, Hamden, CT, USA. We give the solution by Kandall.



The bisector of $\angle BAC$ is AO. Let α, β, γ, θ, and φ be angles as shown in the diagram. Since

$$\frac{\sin\theta}{\sin\alpha} \;=\; \frac{AO}{OE} \;=\; \frac{AO}{OD} \;=\; \frac{\sin\varphi}{\sin\alpha}\,,$$

we have $\sin\theta = \sin\varphi$. Thus, either $\theta = \varphi$, or θ and φ are supplementary.

(a) If $\theta = \varphi$, then $2\beta + \gamma = \beta + 2\gamma$. Consequently, $\beta = \gamma$ and $\triangle ABC$ is isosceles.

(b) If θ and φ are supplementary, then $(2\beta + \gamma) + (\beta + 2\gamma) = 180°$. It follows that $\beta + \gamma = 60°$. Hence, $\angle BAC = 60°$.

4. The real-valued function f is defined for $0 \le x \le 1$, and satisfies $f(0) = 0$, $f(1) = 1$, and

$$\frac{1}{2} \le \frac{f(z) - f(y)}{f(y) - f(x)} \le 2\,,$$

for all $0 \le x < y < z \le 1$ with $z - y = y - x$. Prove that

$$\frac{1}{7} \le f\left(\frac{1}{3}\right) \le \frac{4}{7}\,.$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution.

Let $f\left(\frac{1}{3}\right) = a$ and $f\left(\frac{2}{3}\right) = b$. Setting $x = 0$, $y = \frac{1}{3}$, and $z = \frac{2}{3}$, we get

$$\frac{1}{2} \le \frac{b-a}{a} \le 2\,. \qquad (1)$$

Setting $x = \frac{1}{3}$, $y = \frac{2}{3}$, and $z = 1$, we obtain

$$\frac{1}{2} \le \frac{1-b}{b-a} \le 2\,. \qquad (2)$$

Suppose that $a < 0$. From (1), we deduce that $b - a < 0$; then $b < 0$, and hence $1 - b > 0$. Then, from (2), we get $b - a > 0$, a contradiction. Thus, $a > 0$.

Using (1), we deduce that $b - a > 0$. Then (1) can be rewritten as

$$b \le 3a \le 2b\,, \qquad (3)$$

and (2) can be rewritten as

$$1 + 2a \le 3b \le 2 + a\,, \qquad (4)$$

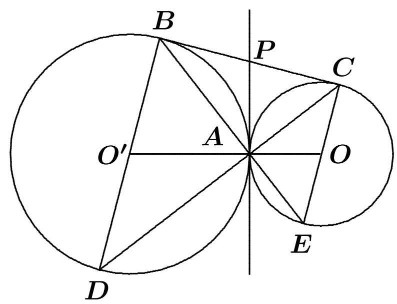
From the inequalities on the right in (3) and (4), we get $2 + a \ge 3b \ge \frac{9}{2}a$, and then $a \le \frac{4}{7}$. From the inequalities on the left, we get $1+2a \le 3b \le 9a$, and then $a \ge \frac{1}{7}$. Thus, $\frac{1}{7} \le a \le \frac{4}{7}$, and we are done.

Next we turn to solutions to problems of the Finnish High School Mathematics Competition 2000 given in [2003 : 436].

1. Two circles touch each other externally at A. A common tangent touches one circle at B and the other at C ($B \ne C$). The segments BD and CE are diameters of the circles. Prove that D, A, and C are collinear.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We first give Bradley's write-up.

Let the tangent at A meet BC at P. Then $PA = PB = PC$ (since these three segments are tangents from the external point P). Hence, P is the centre of the circle BAC. Thus, BC is a diameter of this circle, and $\angle BAC = 90°$. Similarly, $\angle BAD = 90°$, since BD is a diameter of the circle BAD. Therefore,



$$\begin{aligned} \angle DAC &= \angle BAD + \angle BAC \\ &= 90° + 90° \;=\; 180° \,. \end{aligned}$$

Hence, D, A, C are collinear.

Next we give the presentation by Bataille.

Consider the homothety h with centre A which transforms the circle ACE into the circle ABD. Since the lines CE and BD are both perpendicular to the line BC, we have $CE \parallel BD$. In addition, CE and BD pass through the centres O and O' of ACE and ABD (respectively) and $h(O) = O'$; whence, the image of the line CE under h is the line BD. Therefore $h(C)$ is a point on the line BD and on the circle ABD. But $h(C)$ cannot be B (since C, A, B are not collinear); thus, we must have $h(C) = D$, and D, A, C are collinear.

2. Prove that the integer part of $(3+\sqrt{5})^n$ is odd for every positive integer n.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution.

Let n be a positive integer. Let

$$S_n \;=\; \left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n \,.$$

Using the Binomial Theorem, we have

$$S_n \;=\; \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} \left(\left(\sqrt{5}\right)^k + \left(-\sqrt{5}\right)^k \right) \;=\; 2 \sum_{k\geq 0} \binom{n}{2k} 3^{n-2k} 5^k \,,$$

with the usual convention that $\binom{n}{p} = 0$ for $p > n$. Then S_n is an even integer.

Moreover, since $0 < 3 - \sqrt{5} < 1$, we deduce that $0 < \left(3-\sqrt{5}\right)^n < 1$. It follows that $S_n - 1 < \left(3+\sqrt{5}\right)^n < S_n$. Thus, $\left\lfloor \left(3+\sqrt{5}\right)^n \right\rfloor = S_n - 1$, which is odd.

3. Determine all positive integers n such that $n! > \sqrt{n^n}$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Chen and Wang.

Clearly, $n! = \sqrt{n^n}$ for $n = 1$ and $n = 2$. We show by induction that $n! > \sqrt{n^n}$, or equivalently, $(n!)^2 > n^n$, for all $n \geq 3$. This is clearly true for $n = 3$. Suppose $(n!)^2 > n^n$ for some $n \geq 3$. Then

$$\big((n+1)!\big)^2 \;=\; (n+1)^2(n!)^2 \;>\; (n+1)^2 n^n\,. \tag{1}$$

Since $(\frac{n+1}{n})^n = (1+\frac{1}{n})^n < e \leq 3 < n+1$, we have $n^n > (n+1)^{n-1}$. Hence,

$$(n+1)^2 n^n \;>\; (n+1)^{n+1}\,. \tag{2}$$

From (1) and (2), it follows that $\big((n+1)!\big)^2 > (n+1)^{n+1}$, completing the induction.

4. There are seven points in the plane, no three of which are collinear. Every point is joined to every other by either a blue or a red line segment. Prove that there are at least four monochromatic triangles in the figure.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

More generally, we will prove that if $n \geq 3$ points are pairwise joined by a red or a blue line segment, then the minimum number of monochromatic triangles is $f(n)$, with:

$$f(2a+2) \;=\; \frac{a(a-1)(a+1)}{3} \quad\text{and}\quad f(2a+1) \;=\; \left\lceil \frac{a(2a+1)(a-2)}{6} \right\rceil,$$

(where $\lceil x \rceil$ denotes the least integer greater than or equal to x).

Let A_1, A_2, $\ldots$, A_n be the given points. For i, j, k pairwise distinct in $\{1, 2, \ldots, n\}$, let $p_i(j,k) = 2$ if the line segments A_iA_j and A_iA_k have the same color, and let $p_i(j,k) = -1$ otherwise. Note that if the triangle $A_iA_jA_k$ is monochromatic then $p_i(j,k) + p_j(k,i) + p_k(i,j) = 6$, and $p_i(j,k) + p_j(k,i) + p_k(i,j) = 0$ otherwise.

Let $p_i = \sum\limits_{\substack{1 \leq j < k \leq n \\ j,\, k \neq i}} p_i(j,k)$ and $p = \sum\limits_{i=1}^{n} p_i$. Let b and r denote the number of blue and red monochromatic triangles, respectively.

Since each pair of adjacent line segments belongs to only one triangle, we then have

$$p \;=\; 6(b+r)\,. \tag{1}$$

On the other hand, for a given i, let k be the number of red line segments with end-point A_i. Thus, there are $n-1-k$ blue segments with end-point A_i.

We then have:

$$\begin{aligned} p_i &= 2\binom{k}{2} + 2\binom{n-1-k}{2} - \binom{k}{1}\binom{n-1-k}{1} \\ &= n^2 - 3nk - 3n + 3k^2 + 3k + 2 \\ &= 3\left(k - \frac{n-1}{2}\right)^2 + \frac{(n-5)(n-1)}{4}\,. \end{aligned} \qquad (2)$$

Case (i). $n = 2a+2$ is even.

From (2), we see that $p_i \geq (3 + (n-5)(n-1))/4 = a(a-1)$ for each i. Using (1), it follows that $b + r \geq \frac{1}{6}na(a-1) = a(a+1)(a-1)/3$. Since $f(2a+2)$ and $a(a+1)(a-1)/3$ are integers, we have

$$f(2a+2) \;\geq\; \frac{a(a+1)(a-1)}{3}\,.$$

To establish the opposite inequality, we consider the complete graph whose vertices are the $2a+2$ given points. Now colour the edge A_iA_j red if $i \equiv j \pmod 2$ and blue otherwise. It is easy to verify that there are exactly $2\binom{a+1}{3} = a(a+1)(a-1)/3$ monochromatic [*Ed:* actually red] triangles, which proves that $f(2a+2) \leq a(a+1)(a-1)/3$.

Thus, $f(2a+2) = a(a+1)(a-1)/3$, as claimed.

Case (ii). $n = 2a+1$ is odd.

From (2), we see that $p_i \geq (n-5)(n-1)/4 = a(a-2)$, for each i. Using (1), it follows that $b + r \geq \frac{1}{6}na(a-2) = a(2a+1)(a-2)/6$. Since $f(2a+1)$ is an integer, we deduce that

$$f(2a+1) \;\geq\; \left\lceil \frac{a(2a+1)(a-2)}{6} \right\rceil .$$

If a is even, this inequality is equivalent to

$$f(4m+1) \;\geq\; \frac{2m(4m+1)(m-1)}{3}\,, \qquad (3)$$

which is an integer. If a is odd, it is equivalent to

$$f(4m+3) \;\geq\; \frac{(2m+1)(4m+3)(2m-1)}{6} + \frac{1}{2}\,, \qquad (4)$$

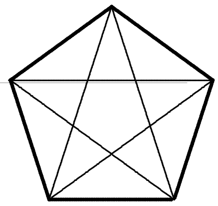
since $(2m+1)(4m+3)(2m-1)/6$ is half an integer.

To establish the opposite inequality, we will again use graph theory terminology.

Case (a). $n = 4m+1$.

Our proof will be by induction on m. First, for $m = 1$, we consider the complete graph K_5 where the edges are coloured as in the diagram below right (the thick lines represent red edges, the thin lines blue).

This graph has no monochromatic triangles, from which we deduce that $f(5) = 0$. Note that in this graph, at each vertex there are the same number of red and blue edges. We call such a graph *balanced*. Now, assume that for a given $m \geq 2$ we have a $K_{4(m-1)+1}$ balanced edge-bicoloured complete graph. Now let us consider a complete graph K_{4m+1}, and colour the edges of that graph in red and blue as follows:



First, construct a balanced $K_{4(m-1)+1}$ for the subgraph whose vertices are A_5, A_6, $\dots$, A_{4m+1} (which is possible by the induction hypothesis). Then, colour the edges of the subgraph having vertices A_1, A_2, A_3, A_4, A_5 as in the K_5 above. Last, for each $i \geq 6$, give the colour of A_5A_i to A_1A_i and A_2A_i, and the opposite one to A_3A_i and A_4A_i. It is easy to verify that the bicolouration of the edges of the K_{4m+1} gives a balanced graph.

It follows that for each $m \geq 1$, we may consider a balanced K_{4m+1}. For such a complete graph, for each i the number of red edges from A_i is equal to $k = 2m$, from which (using (2)) we deduce that $p_i = (n-5)(n-1)/4$. From (1), we then have $b + r = \frac{1}{6}n(n-5)(n-1)/4 = 2m(4m+1)(m-1)/3$. Thus, $f(4m+1) \leq 2m(4m+1)(m-1)/3$.

Combining this with (3), we are done.

Case (b). $n = 4m + 3$.

Let us colour the edges of the complete graph K_{4m+3} as follows: First colour the edges of the subgraph K_{4m+1} having vertices A_3, A_4, $\dots$, A_{4m+3} to obtain a balanced graph as above.

Then, for each $i \geq 4$, give the colour of A_3A_i to A_1A_i and the opposite one to A_2A_i. Now give colour blue to A_1A_2, and red to A_1A_3 and A_2A_3. It is easy to verify that for each $i \neq 3$, we have exactly $k = 2m + 1$ red edges from A_i, so that (using (2)) $p_i = (n-5)(n-1)/4$. Moreover, the number of red edges from A_3 is $2m$, so that $p_3 = 3 + \big((n-5)(n-1)\big)/4$. It follows that $p = 3 + \big(n(n-5)(n-1)\big)/4$ and

$$b + r \;=\; \frac{(2m+1)(4m+3)(2m-1)}{6} + \frac{1}{2}\,.$$

Thus,

$$f(4m+3) \;\leq\; \frac{(2m+1)(4m+3)(2m-1)}{6} + \frac{1}{2}\,,$$

and we are done.

For the special case $n = 7$, we have $f(7) \geq \left\lceil \dfrac{3(2\times 3+1)(3-2)}{6} \right\rceil = 4$, which solves the given problem.

That completes this number of the *Corner* and the solutions on file for the 2003 numbers. Send me your nice solutions and generalizations.