SKOLIAD No. 88

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Please send your solutions to the problems in this issue by ***February 1, 2006***. A copy of MATHEMATICAL MAYHEM Vol. 4 will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

In this issue we feature the 2004 W.J. Blundon Mathematics Contest, for which I thank Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.

21[ième] Concours de Mathématiques W.J. Blundon
Commandité par la SMC et le département de mathématiques de l'Université Mémorial, 18 Février 2004

1. Un fermier dépense exactement $100\,\$$ pour acheter 100 animaux. Une vache coûte $10\,\$$, un mouton $3\,\$$ et un cochon 50 cents. Combien de chaque sorte a-t-il acheté ?

2. Montrer que si un nombre à trois chiffres est divisible par 3, alors la somme de ses chiffres est divisible par 3.

3. Considérer les points $A(1,0)$, $B(3,0)$, $C(3,5)$ et $D(1,4)$. Trouver l'équation de la droite passant par l'origine qui coupe le quadrilatère $ABCD$ en deux parties de même aire.

4. Trouver toutes les solutions réelles de l'équation $1+x+x^2+x^3 = x^4+x^5$.

5. Trouver l'aire exacte de l'octogone régulier formé en coupant des triangles rectangles isocèles congrus des coins d'un carré dont les côtés mesurent une unité.

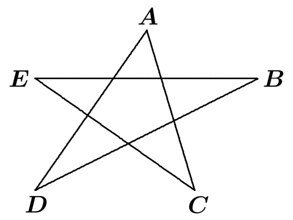
6. Si A, B et C sont les angles d'un triangle, montrer que

$$\cos C \;=\; \sin A \sin B - \cos A \cos B\,.$$

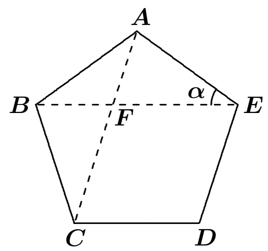
7. Si $a+b+c=0$ et $abc=4$, trouver $a^3+b^3+c^3$.

8. (a) Si $\log_{10} 2 = a$ et $\log_{10} 3 = b$, trouver $\log_5 12$.
(b) Résoudre $x^{\log_{10} x} = 100x$.

9. Dans la figure suivante, trouver la somme des angles A, B, C, D et E.



10. Soit $ABCDE$ le pentagone régulier de côté 1. La longueur de BE est τ, et l'angle FEA est α. Trouver τ et $\cos\alpha$.



The 21st W.J. Blundon Mathematics Contest
Sponsored by the CMS and the Mathematics Department at Memorial University, February 18, 2004

1. A farmer spent exactly \$100 to buy 100 animals. Cows cost \$10, sheep \$3 and pigs 50 cents each. How many of each did he buy?

2. Show that if a three-digit number is divisible by 3, then the sum of its digits is divisible by 3.

3. Consider the points $A(1,0)$, $B(3,0)$, $C(3,5)$, and $D(1,4)$. Find an equation of the line through the origin that divides the quadrilateral $ABCD$ into two parts of equal area.

4. Find all real solutions to the equation $1 + x + x^2 + x^3 = x^4 + x^5$.

5. Find the exact area of the regular octagon formed by cutting equal isosceles right triangles from the corners of a square with sides of length one unit.

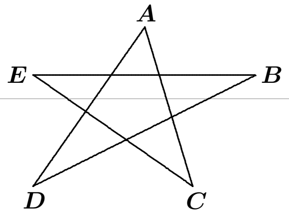
6. If A, B, and C are angles of a triangle, prove that

$$\cos C \;=\; \sin A \sin B - \cos A \cos B\,.$$

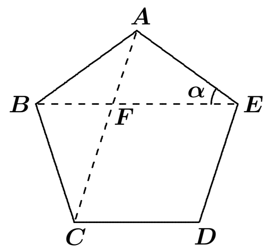
7. If $a + b + c = 0$ and $abc = 4$, find $a^3 + b^3 + c^3$.

8. (a) If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, find $\log_5 12$.
(b) Solve $x^{\log_{10} x} = 100x$.

9. In the figure below, find the sum of the angles A, B, C, D, and E.



10. Let $ABCDE$ be a regular pentagon with each side of length 1. The length of BE is τ, and the angle FEA is α. Find τ and $\cos\alpha$.



Next we give the solutions to the South African Interprovincial Mathematics Olympiad 2004 [2005 : 65–68].

South African Interprovincial Mathematics Olympiad 2004

Team Paper: Juniors, 60 minutes allowed

1. $(*)$ Five bags of rice are weighed two at a time, in all possible combinations. The ten weights are 72, 73, 76, 77, 79, 80, 81, 83, 84, and 87. What are the weights of the five bags?

Solution by the editor.

First note that no two bags can have the same weight, because all of the weights given in the problem are different. For example, if bag #1 had the same weight as bag #2, then the total weight of bag #1 and bag #3 would equal that of bag #2 and bag #3.

Let a, b, c, d, and e represent the weights of the bags from lightest to heaviest. Thus, $a < b < c < d < e$. The lightest pair of bags is a and b, and the next lightest pair is a and c. Therefore, $a + b = 72$ and $a + c = 73$. Similarly, we must have $d+e = 87$ and $c+e = 84$. Since $a+c = (a+b)+1$ and $d + e = (c + e) + 3$, we have $c = b + 1$ and $d = c + 3$.

The next smallest sum, after $a+b$ and $a+c$ is either $a+d$ or $b+c$. Therefore, $\{b{+}c, a{+}d\} = \{76, 77\}$. Suppose that $b{+}c = 76$. Together with $a+b = 72$ and $a+c = 73$, this yields $a = 34.5$, $b = 37.5$, and $c = 38.5$. Since $a+d = 77$, we also have $d = 42.5$, which contradicts $d = c+3$. Therefore, we must have $b+c = 77$ and $a+d = 76$. A similar argument then yields $a = 34$, $b = 38$, $c = 39$, $d = 42$. From this, we easily obtain $e = 45$. A quick check shows that this is, indeed, a solution.

Hence, the five bags weigh 34, 38, 39, 42, and 45.

There were three solutions submitted which found the correct answer but assumed that the weights were all integers. It is true that the weights turn out to be integers, but this cannot be assumed in advance; for example, suppose the original ten weights were 69, 72, 73, 74, 75, 77, 78, 79, 82, *and* 87.

2. ($*$) Delete 60 digits from the number $1\,2\,3\,4\,5\,6\,\ldots\,38\,39\,40$ in such a way as to make the resulting number as small as possible.

[*Ed:* Two different interpretations were given to the question. In the first solution below, the aim was the absolute lowest number possible and thus could start with 0, while in the second, the lowest number had to have 11 digits and thus start with a 1.]

I. *Solution par Marianella Ouellet, catégorie* 12*, Collège Montmorency, Laval, QC.*

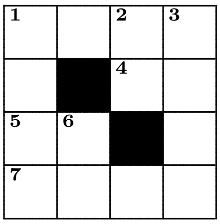
Tout d'abord, il y a 71 chiffres dans cette série. On sait alors que si on enlève 60 chiffres de la liste, on obtient un nombre de 11 chiffres de long. Ensuite, on doit garder tous les zéros car ceux-ci ne peuvent que rapetisser le résultat final. De plus, on doit éliminer tous les chiffres de 1 jusqu'à 30 en excluant évidemment les zéros, car malgré que l'on élimine des 1, on se rend compte que ceux-ci donneraient une valeur au zéros qui les suivent. Pour finir, il s'agit de prendre les plus petits chiffres dans l'ordre et on obtient un résultat de 00012333330.

II. *Solution by Geoffrey Siu, grade* 12*, London Central Secondary School, London, ON.*

There are $9+31\times 2 = 71$ digits. We will have $71-60 = 11$ digits left. For the smallest number, we want the first digit to be as small as possible; the first digit must be 1. Then we want as many 0s as possible. We can have $10\,00x\,xxx\,xxx$ by using the 0s from 10, 20, and 30. But we cannot have another 0 (there are not enough numbers). Now, we use 1, 2, and 3 (from 31, 32, and 33), more 3s (from the numbers 34 through 39), then 0 (from 40), obtaining the smallest possible digit at each step. The smallest number is therefore 10001233330.

Also solved by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON. One incorrect solution was submitted.

3. Solve the crossnumber puzzle:



Across	**Down**
1. Cube of a prime	1. Square of a prime
4. Square	2. Three times cube root of 1 Across
5. Square	3. Square of a prime
7. Cube	6. Twice cube root of 7 Across

Solution by Geoffrey Siu, grade 12, London Central Secondary School, London, ON, modified by the editor.

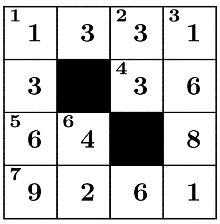
Since $10^3 = 1000$ and $22^3 = 10648 > 9999$, then #1 Across is the cube of one of 11, 13, 17, and 19. Since the third digit of #1 Across is the first digit of #2 Down, only $11^3 = 1331$ and $19^3 = 6859$ are acceptable for #1 Across. But from #4 Across and the fact that there are no squares starting with 7, we see that #1 Across is 1331. Then #2 Down is 33 and #4 Across is 36.

Now #3 Down is the square of a prime and starts with 16. The only possibility is $41^2 = 1681$.

Since #7 Across is a cube between 10^3 and 21^3 ending with 1, it is either $11^3 = 1331$ or $21^3 = 9261$. But #6 Down is an even number. Hence, we conclude that #7 Across is 9261. Thus, #6 Down is 42.

Then #5 Across is a 2-digit square ending in 4, which leaves us only one choice, namely 64.

Now #1 Down has the form 1?69 and it is the square of a prime. By trial and error, #1 Down must be $37^2 = 1369$.



Also solved by Bobby Xiao, grade 10, Walter Murray Collegiate Institute, Saskatoon, SK. One solver simply supplied the answer.

4. Find the sum of the digits of $10^{2004} - 2004$.

Solution by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON.

Note that 10^{2004} has 2005 digits. When we substract 2004 from 10^{2004}, we get a number with 2004 digits. This number has the form $999\ldots997996$. Because this number has 2004 digits, the number of nines in the beginning of the number is $2004 - 4 = 2000$. Thus, the sum of the digits of this number is $9 \times 2000 + 7 + 9 + 9 + 6 = 18031$.

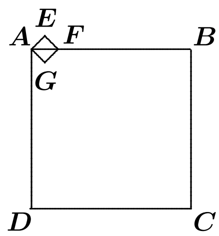
Also solved by Bobby Xiao, grade 10, Walter Murray Collegiate Institute, Saskatoon, SK; and Geoffrey Siu, grade 12, London Central Secondary School, London, ON.

5. An urn contains 100 balls of different colours: namely, 10 white, 10 black, 12 yellow, 14 blue, 24 green, and 30 red. What is the minimum number of balls that must be drawn from the urn without looking if you want to be certain that at least 15 of the balls drawn are of the same colour?

Identical solutions by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON; Bobby Xiao, grade 10, Walter Murray Collegiate Institute, Saskatoon, SK; and Geoffrey Siu, grade 12, London Central Secondary School, London, ON.

In the worst case, you draw all the white, black, yellow, and blue balls, 14 of the green balls, and 14 of the red balls, before you get 15 of any colour. Then you have drawn $10+10+12+14+14+14 = 74$ balls. There are then only red and green balls left, and any draw will give you 15 of one colour. Hence, 75 balls drawn guarantees 15 of the same colour.

6. In the diagram at right, square $ABCD$ has side 24 cm and square $AEFG$ has side 2 cm. What is the length of CE in cm?



I. *Similar solutions by Bobby Xiao, grade 10, Walter Murray Collegiate Institute, Saskatoon, SK; and Geoffrey Siu, grade 12, London Central Secondary School, London, ON.*

Extend EG to cross DC at I. Let H be the intersection of EG and AB. Since EG and AF are diagonals of the square $AEFG$ with F on AB, we have $EG \perp AB$. Since $AB \parallel DC$, we see that $EG \perp DC$ and $\triangle EIC$ has a right angle at I. We have $EH = AH = HF = HG = \sqrt{2}$ (since each side of $AEFG$ has length 2). Thus, we have $EI = EH + HI = 24 + \sqrt{2}$. Now $HBCI$ is a rectangle; whence, $CI = HB = AB - AH = 24 - \sqrt{2}$. Applying the Pythagorean Theorem to $\triangle EIC$, we have

$$CE = \sqrt{\left(24+\sqrt{2}\right)^2 + \left(24-\sqrt{2}\right)^2} = 34 \text{ cm}.$$

II. *Solution by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON.*

Applying the Pythagorean Theorem to $\triangle ADC$, we find that $AC = \sqrt{AD^2 + DC^2} = 24\sqrt{2}$ cm. Since AF is a diagonal of the square $AEFG$, we have $\angle FAG = 45^\circ$. Since AC is the diagonal of the square $ABCD$, we get $\angle BAC = 45^\circ$, and G lies on AC. Hence, $\triangle EAC$ has a right angle at A. Thus, by the Pythagorean Theorem in $\triangle EAC$, we have $EC = \sqrt{AC^2 + EA^2} = \sqrt{\left(24\sqrt{2}\right)^2 + 2^2} = 34$ cm.

7. All the positive integers, starting with 1, are written in order, namely,

$$12345678910111213141516\ldots$$

Find the digit appearing in the 206 788$^{\text{th}}$ position.

Solution by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON.

There are 9 one-digit integers, 90 two-digit integers, 900 three-digit integers, 9000 four-digit integers. These numbers are the integers from 1 to 9999, and the total number of digits in them is $9 \times 1 + 90 \times 2 + 900 \times 3 + 9000 \times 4 = 38889$.

After that come the five-digit numbers. Between positions 38889 and 206788, there are $\left\lfloor \frac{206788 - 38889}{5} \right\rfloor = 33579$ complete five-digit numbers. The last one of them will be $9999 + 33579 = 43578$. Number 43579 will start in position 206785. Thus, 7 will be in position 206788.

There were two solutions submitted which had small errors in them.

8. How many times does the number 2 appear when the product

$$1002 \cdot 1003 \cdot 1004 \cdots 2004$$

is expanded into its prime factors?

Solution by the editor.

Since the given product can be expressed as $\frac{2004!}{1001!}$, it is sufficient to calculate the number of times that the prime 2 appears in 2004! and subtract the number of times it appears in 1001!.

Let n be any positive integer. We will now examine the number of times a prime p appears in $n!$. We note first that every multiple of p between 1 and n will contribute a factor of p. This generates $\left\lfloor \frac{n}{p} \right\rfloor$ factors (where $\lfloor x \rfloor$ is the greatest integer not exceeding x). However, multiples of p^2 also contribute a second factor of p. This generates $\left\lfloor \frac{n}{p^2} \right\rfloor$ more factors of p. Similarly, by considering the multiples of p^3 we get $\left\lfloor \frac{n}{p^3} \right\rfloor$ more factors of p. And so on. Thus the number of factors of the prime p in $n!$ is given by

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{p^k} \right\rfloor + \cdots$$

(The above summation has only finitely many non-zero terms, since p^k will eventually exceed n.)

Using this result, we see that the number of 2s in $2004!$ is

$$\left\lfloor \frac{2004}{2} \right\rfloor + \left\lfloor \frac{2004}{4} \right\rfloor + \left\lfloor \frac{2004}{8} \right\rfloor + \cdots + \left\lfloor \frac{2004}{1024} \right\rfloor$$
$$= \; 1002 + 501 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 \; = \; 1997\,.$$

Similarly, the number of 2s in $1001!$ is

$$\left\lfloor \frac{1001}{2} \right\rfloor + \left\lfloor \frac{1001}{4} \right\rfloor + \left\lfloor \frac{1001}{8} \right\rfloor + \cdots + \left\lfloor \frac{1001}{512} \right\rfloor$$
$$= \; 500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 \; = \; 994\,.$$

Thus, the number of 2s in the given product is simply $1997 - 994 = 1003$. (One could also observe that the last 8 terms in the two summations are the same, and could simply be cancelled, leaving a much simpler computation.)

Also solved by Geoffrey Siu, grade 12, London Central Secondary School, London, ON; and Bobby Xiao, grade 10, Walter Murray Collegiate Institute, Saskatoon, SK. One incorrect solution was submitted.

9. In the addition below, digits have been replaced by letters in a one-to-one fashion. Given that $D = 5$, work out the original numbers.

D	O	N	A	L	D
G	E	R	A	L	D
R	O	B	E	R	T

Solution by Bobby Xiao, grade 10, Walter Murray Collegiate Institute, Saskatoon, SK.

We are given that $D = 5$; since $5 + 5 = 10$, we have $T = 0$ with a carry of 1 into the tens column. Now R must be odd, since $2L + 1 = R$. From $O + E = O$ (with a possible carry), we have either $E = 0$ or $E = 9$, depending on whether there is a carry into that column. But $E \neq T = 0$, which means that $E = 9$, and this addition has a carry, as does $N + R = B$. Since $5 + G + 1 = R$ does not carry, we see that $G < 4$ and $R > 6$, since $G \neq 0$. But R is odd; hence, $(G, R) = (1, 7)$ or $(G, R) = (3, 9)$. But $R \neq E = 9$; this implies that $(G, R) = (1, 7)$. Since $2A \neq E = 9$, there must be a carry into the hundreds column. Hence, $2L + 1 = 17$, which gives us $L = 8$, and $2A + 1 = 9$ gives us $A = 4$. Only N, B, and O remain with 2, 3, and 6 left to assign. Now $N + 7 = 10 + B$ simplifies to $N = B + 3$ and $(N, B) = (6, 3)$ is the only fit among the remaining values. Hence, $O = 2$, since it is the only remaining value. The addition was thus:

5	2	6	4	8	5
1	9	7	4	8	5
7	2	3	9	7	0

Also solved by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON; and Geoffrey Siu, grade 12, London Central Secondary School, London, ON.

10. Consider a square having 16 cells each containing a plus sign or a minus sign. Suppose we change all the signs in a given row (or column), doing this several times until the number of minus signs is a minimum. A table that has the property that any such change does not decrease the number of minus signs is called a *minimal table*, and the number of minus signs in a minimal table is called the *characteristic* of the table. Find all possible values of the characteristic.

Solution by Bobby Xiao, grade 10, Walter Murray Collegiate Institute, Saskatoon, SK.

There cannot be more than 2 minus signs in any row or column of a minimal table because if there were 3 or 4 minus signs, then changing all the signs in that row or column would reduce the overall number of minus signs. There cannot be more than 8 minus signs in a minimal table, since there are only 4 rows and each can hold at most 2 minus signs. Thus, the largest possible characteristic is 8. Examples of minimal tables with characteristics from 0 to 8 are shown below:

+	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

-	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

-	+	+	+
+	-	+	+
+	+	+	+
+	+	+	+

-	+	+	+
+	-	+	+
+	+	-	+
+	+	+	+

-	+	+	+
+	-	+	+
+	+	-	+
+	+	+	-

-	+	+	+
-	-	+	+
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+	+	+	-

-	+	+	+
-	-	+	+
+	-	-	+
+	+	+	-

-	+	+	+
-	-	+	+
+	-	-	+
+	+	-	-

-	+	+	-
-	-	+	+
+	-	-	+
+	+	-	-

Also solved by Alexander Remorov, grade 9, Waterloo Collegiate Institute, Waterloo, ON. One incorrect solution was submitted.

That brings us to the end of another Skoliad.