

# Misère Games

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## Abstract.

The theory of last-player-winning counter-pickup games is well known. See [1] and [2]. The corresponding *misère* games in which the last player loses are less well understood. In this note, we define a special class of combinatorial games and find the winning strategies for all composite games with these special games as components. In the first section we recall the method of using Nim values of component games to solve a composite game. In the second section, we define *special* games and find winning strategies for *misère* games.

## Section 1: Terminology

**Definition.** A finite impartial game  $G$  played under normal rules of play is called a *regular* game. This means that

1. two players alternate moving,
2. there is no infinite sequence of moves,
3. both players have the same moves available, and
4. the winner is the last player to make a move.

Such a game can be thought of as a directed acyclic graph. Each *vertex* of the graph corresponds to a position in the game and each *directed edge* corresponds to a move. The *followers* of a vertex are those positions joined to it by an outgoing edge. We will briefly say that  $G$  is a regular impartial game.

**Nim Values.** The minimum excluded value, denoted *mex*, of a finite set of non-negative integers is the least non-negative integer not in the set. For example,  $\text{mex}\{1, 2, 4, 0\} = 3$ ,  $\text{mex}\{2, 4, 5\} = 0$ ,  $\text{mex}\{\} = 0$ . The *Nim value* of a position, denoted by  $g(n)$ , is the *mex* of the Nim values of its followers. A position with no followers (that is, a *terminal* position) has Nim value 0. It is easy to see that the winning strategy is to move to a position with Nim value 0, for then the opponent either has no move at all and loses immediately, or must move to a position with Nim value greater than 0 and hence must eventually lose.

**Composite Games.** *Composite games*, denoted  $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$ , are games that have several components. Two players alternate moves. Each player on his turn selects a component game  $G_i$  in which a legal move can

be made and makes a legal move in that game. The winner is the last player to move. We define the Nim value of the composite game as the Nim sum (denoted by  $\oplus$ ) of the Nim values of each of the component games. The Nim sum is obtained by writing the integers in binary and adding modulo 2 without carrying. For example,  $6 \oplus 3 = 110_{(2)} \oplus 11_{(2)} = 101_{(2)} = 5_{(10)}$ , since, by considering the digits in the summation from left to right, we get  $1 \equiv 1 \pmod{2}$ ,  $1 + 1 \equiv 0 \pmod{2}$ , and  $0 + 1 \equiv 1 \pmod{2}$ .

**Strategy.** The *balanced* positions are those positions whose Nim values are 0. The *unbalanced* positions are those positions whose Nim values are not zero.

If a position is balanced, it will always become unbalanced after the moving player moves. This follows from the definition of mex since if a player moves from  $n_i$  to  $m_i$  in  $G_i$ , then  $g(n_i) \neq g(m_i)$ .

Also, if a position is unbalanced, the moving player can always move to a balanced position. Such a winning move can always be selected from the component  $G_i$  that contributes the left-most 1 in the Nim sum of the component values. This follows from the definition of mex since if  $g(n_i) \geq 1$  in game  $G_i$ , the moving player can move in  $G_i$  to a vertex  $m_i$  having any of the values  $\{0, 1, 2, \dots, g(n_i) - 1\}$ . In particular, the moving player can move to a position  $m_i$  whose value is the sum of the Nim values of the other components. Of course, all terminal positions have a Nim value of  $0 \oplus 0 \oplus \dots \oplus 0 = 0$ , which is balanced.

## Section 2

**Misère version of a game.** The misère version of a regular impartial game  $G_i$  is played by the same rules as  $G_i$  except the loser is the player who makes the last move.

The misère version of a composite game  $G_1 \oplus G_2 \oplus \dots \oplus G_k$  is played by the same rules as  $G_1 \oplus G_2 \oplus \dots \oplus G_k$  except the loser is the last player to move.

**Special Games** Suppose  $G$  is a regular impartial game. We say that  $G_i$  is *special* if, for each position  $n$  in  $G$ , when  $g(n) = 0$ , we have either (i)  $n$  is a terminal position or (ii) there exists a follower  $m$  of  $n$  such that  $g(m) = 1$ .

**Problem 1.** Suppose  $G_1, G_2, \dots, G_k$  are special, regular impartial games. Find a strategy for playing the misère version of  $G_1 \oplus G_2 \oplus \dots \oplus G_k$ .

**Solution.** Let  $(n_1, n_2, \dots, n_k)$  denote an arbitrary position in the composite game  $G_1 \oplus G_2 \oplus \dots \oplus G_k$ . We will first define the balanced positions.

- A. If each  $g(n_i) \in \{0, 1\}$ , then  $(n_1, n_2, \dots, n_k)$  is balanced if and only if  $g(n_1) \oplus g(n_2) \oplus \dots \oplus g(n_k) = 1$ .
- B. If at least one  $g(n_i) \notin \{0, 1\}$ , then  $(n_1, n_2, \dots, n_k)$  is balanced if and only if  $g(n_1) \oplus g(n_2) \oplus \dots \oplus g(n_k) = 0$ .

Let  $B, U$  denote the balanced and unbalanced positions respectively.

We note that all terminal positions, which we denote  $0$ , are unbalanced. We will prove the following which we have illustrated in Figure 1.

- (1) If  $(n_1, n_2, \dots, n_k)$  is balanced, then all moves must be to an unbalanced position.
- (2) If  $(n_1, n_2, \dots, n_k)$  is unbalanced and non-terminal, then there exists a move to a balanced position.

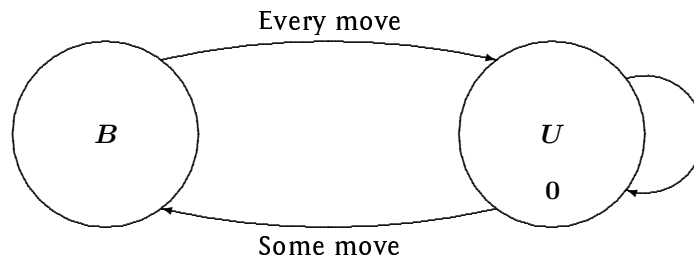


Figure 1

From (1) and (2) it follows that if  $(n_1, n_2, \dots, n_k)$  is the initial position in the game, then

- (a) if  $(n_1, n_2, \dots, n_k)$  is balanced, the first player to move will lose if the opposing player plays perfectly.
- (b) if  $(n_1, n_2, \dots, n_k)$  is unbalanced, then the first player to move will win with perfect play.

We now prove (1) and (2).

*Proof of (1).* For the balanced position  $(n_1, n_2, \dots, n_k)$ , we consider two cases.

**Case (a).**  $g(n_i) \in \{0, 1\}$  for all  $i$ .

Since the position is balanced, we have  $g(n_1) \oplus g(n_2) \oplus \dots \oplus g(n_k) = 1$ . Without loss of generality, we may assume the player to move chooses to make a move in game  $G_1$ , which must be non-terminal, of course.

If  $g(n_1) = 0$ , then, by the definition of  $\text{mex}$ , the player to move must move to  $m_1$  with  $g(m_1) = 1$  or  $g(m_1) \geq 2$ . In either case, the new position,  $(m_1, n_2, n_3, \dots, n_k)$ , is unbalanced.

If  $g(n_1) = 1$ , then, by the definition of  $\text{mex}$ , the player to move must move to  $m_1$  with  $g(m_1) = 0$  or  $g(m_1) \geq 2$ . In either case, the new position,  $(m_1, n_2, n_3, \dots, n_k)$ , is unbalanced.

**Case (b).**  $g(n_i) \notin \{0, 1\}$  for some  $i$ .

Then  $g(n_1) \oplus g(n_2) \oplus \dots \oplus g(n_k) = 0$ , which implies there must also be a  $j \neq i$  such that  $g(n_j) \notin \{0, 1\}$ . Now after the next move, there must

still exist a game  $G_j$  such that  $g(n_j) \notin \{0, 1\}$ . By the definition of mex, after this next move it will be impossible for  $g(\overline{n_1}) \oplus g(\overline{n_2}) \oplus \dots \oplus g(\overline{n_k}) = 0$  where  $(\overline{n_1}, \overline{n_2}, \dots, \overline{n_k})$  is the new position. Therefore,  $(\overline{n_1}, \overline{n_2}, \dots, \overline{n_k})$  is unbalanced.

*Proof of (2).* For the unbalanced non-terminal position  $(n_1, n_2, \dots, n_k)$ , we consider the two cases.

**Case (a).**  $g(n_i) \notin \{0, 1\}$  for some  $i$ .

- (i) Only one  $g(n_i) \notin \{0, 1\}$ . Since  $g(n_i) \geq 2$ , by the definition of mex, the player to move can move to an  $m_i$  such that  $g(m_i) = 0$  and move to an  $\overline{m_i}$  such that  $g(\overline{m_i}) = 1$ . This easily implies that he can move to a balanced position.
- (ii) Two or more  $g(n_i) \notin \{0, 1\}$ . By the definition of mex, the player to move (as in Bouton's Nim) moves to a position  $(\overline{n_1}, \overline{n_2}, \dots, \overline{n_k})$  such that  $g(\overline{n_1}) \oplus g(\overline{n_2}) \oplus \dots \oplus g(\overline{n_k}) = 0$ , which is a balanced position.

**Case (b).**  $g(n_i) \in \{0, 1\}$  for all  $i$ .

Since the position  $(n_1, n_2, \dots, n_k)$  is unbalanced, it follows that  $g(n_1) \oplus g(n_2) \oplus \dots \oplus g(n_k) = 0$ . Now, since  $(n_1, n_2, \dots, n_k)$  is non-terminal, let  $n_i$  be a non-terminal vertex in a game  $G_i$ . If  $g(n_i) = 1$ , by the definition of mex, the player to move can move to  $m_i$  with  $g(m_i) = 0$ , which balances the game. If  $g(n_i) = 0$ , by the definition of a special game, the player to move can move to  $m_i$  with  $g(m_i) = 1$ , which again balances the game. ■

## References

- [1] Berlekamp, Conway, and Guy, *Winning Ways*, Academic Press, New York, 1982.
- [2] Richard K. Guy, *Fair Game*, 2nd ed., COMAP, New York, 1989.

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