Pólya's Paragon

Fun With Numbers (Part 4)

Shawn Godin

Last issue we introduced the idea of modular arithmetic, and we looked at using digital sums to check calculations. The digital sum of a number is actually a single digit congruent to the number modulo 9. For example, from last issue, $43\,658\,912 \equiv 38 \equiv 11 \equiv 2 \pmod 9$. When we check a calculation using digital sums, we are really checking whether our answer is correct modulo 9.

How can we show this congruence between a number and its digital sum modulo 9? It becomes rather simple when we remember that an n–digit number $d_{n-1}d_{n-2}\ldots d_2d_1d_0$ is really the number

$$d_{n-1} \times 10^{n-1} + d_{n-2} \times 10^{n-2} + \cdots + d_2 \times 10^2 + d_1 \times 10 + d_0 .$$

Now we simply have to notice that $10 \equiv 1 \pmod 9$, which implies that $10^k \equiv 1 \pmod 9$ for any k. Hence,

$$\begin{aligned} & d_{n-1} \times 10^{n-1} + d_{n-2} \times 10^{n-2} + \cdots + d_2 \times 10^2 + d_1 \times 10 + d_0 \\ \equiv\ & d_{n-1} \times 1^{n-1} + d_{n-2} \times 1^{n-2} + \cdots + d_2 \times 1^2 + d_1 \times 1 + d_0 \\ \equiv\ & d_{n-1} + d_{n-2} + \cdots + d_2 + d_1 + d_0 \pmod 9 . \end{aligned}$$

The process of calculating the digital sum of a number has often been called "*casting out nines*", because when you calculate the digital sum you can ignore multiples of 9. For example, if you were to look at $43\,658\,912$, your first step would be to get rid of the 9s to get $43\,658\,\not{9}12$. Then, since $4+5=9$, $3+6=9$, and $8+1=9$, we get

$$\not{4}\not{3}\ \not{6}\not{5}\not{8}\ \not{9}\not{1}2 .$$

Therefore, the digital sum is 2, and we must have $43\,658\,912 \equiv 2 \pmod 9$. Use this trick to amaze your parents and impress your peers!

This can be turned into a divisibility test, since $9 \mid a$ if and only if $a \equiv 0 \pmod 9$ (why?).

Divisibility by 9: A number is divisible by 9 if and only if its digital sum is divisible by nine.

We can extend this idea a little further. By noting that $10^2 = 100$, we can convert a number from base ten to base hundred quite easily by looking at blocks of two digits starting from the right. For example,

$$\begin{aligned} 43\,658\,912 &= 4 \times 10^7 + 3 \times 10^6 + 6 \times 10^5 + 5 \times 10^4 \\ &\qquad + 8 \times 10^3 + 9 \times 10^2 + 1 \times 10 + 2 \\ &= 43 \times 100^3 + 65 \times 100^2 + 89 \times 100 + 12 \\ &= 43{,}65{,}89{,}12_{100} , \end{aligned}$$

where the commas separate the "digits" and the subscript 100 means we are working in base hundred instead of base ten. Then, since $100 \equiv 1 \pmod{99}$, we can see that the digital sum will be equivalent to the number modulo 99. That is, $43 + 65 + 89 + 12 = 209$ and $2 + 09 = 11$. This implies that $43\,658\,912 \equiv 11 \pmod{99}$.

So what? you say. How often do you want to divide something by 99? The real bonus here is that $43\,658\,912 \equiv 11 \pmod{99}$, which tells us that $43\,658\,912 = 99k + 11$ for some integer k. Since we know that $99 = 9 \times 11$, we see that $99 \equiv 0 \pmod{9}$ and $99 \equiv 0 \pmod{11}$, which gives us

$$43\,658\,912 \;\equiv\; 11\,,$$

using either $\bmod 9$ or $\bmod 11$. This means, in effect, that we have come up with one test that tests for divisibility by *both* 9 and 11.

We can use the structure of our number system to come up with other divisibility rules. The next two should be well known:

Divisibility by 2: A number is divisible by 2 if and only if its last digit is 0, 2, 4, 6, or 8.

Divisibility by 5: A number is divisible by 5 if and only if its last digit is either 0 or 5.

We can justify each of these two rules by noting that $10 = 2 \times 5$. Thus, $10 \equiv 0 \pmod{2}$ and $10 \equiv 0 \pmod{5}$. As a result, we have

$$d_{n-1}d_{n-2}\ldots d_2 d_1 d_0 \;\equiv\; d_0\,,$$

using either $\bmod 2$ or $\bmod 5$. We can extend this idea by noting that $4 = 2^2$. Then $100 = 10^2 \equiv 0 \pmod{4}$. Similarly, for $25 = 5^2$ we get $100 = 10^2 \equiv 0 \pmod{25}$. Thus,

$$\begin{aligned} & d_{n-1}d_{n-2}\ldots d_2 d_1 d_0 &\equiv&\; d_1 \times 10 + d_0 \pmod{4} \\ \text{and}\quad & d_{n-1}d_{n-2}\ldots d_2 d_1 d_0 &\equiv&\; d_1 \times 10 + d_0 \pmod{25}\,. \end{aligned}$$

From this we get

Divisibility by 4: A number is divisible by 4 if and only if the two-digit number formed by its last two digits is divisible by 4.

Divisibility by 25: A number is divisible by 25 if and only if the two-digit number formed by its last two digits is divisible by 25 (that is, 00, 25, 50, or 75).

This should be enough to get you going. Here are some things to try for homework.

1. Modify the divisibility test for 9 to get another test for 3.

2. Modify the divisibility test for 9 to get another test for 11. *Hint:* Note that $10 \equiv -1 \pmod{11}$. (Why?)

3. Construct divisibility tests for 2^n and 5^n for any integer $n > 1$.

4. Develop divisibility tests for 101, 1001, 10001, $\ldots$ and 999, 9999, 99999, $\ldots$, and see what else comes out of it.

One last note: the divisibility rule for a composite number is just a combination of the rules for the prime powers which are factors of the number. For example, to test divisibility by 12, you would just test for divisibility by 3 and 2^2, since a number is divisible by 12 if and only if it is divisible by both 3 and 2^2.

Have a great summer! See you again in September!