Problem of the Month

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Problem (2004 United Kingdom Mathematics Trust Junior Math Olympiad) At a summer camp, five students, called A, B, C, D, and E, each take part in five events, called V, W, X, Y, and Z. In each competition scores of 5, 4, 3, 2, and 1 are awarded for 1st, 2nd, 3rd, 4th, and 5th, respectively. There are no ties. Student A scores a total of 24 points, student C scores the same in each of four events, student D scores 4 in competition V, and student E scores 5 in W and 3 in X. Surprisingly, their overall positions are in alphabetical order. There are no ties in the final standings. Show that this information is enough to find all the scores, and that there is only one solution.

Trying to solve this problem is a good exercise in logic, as well as numerical manipulation. There is no high-level mathematics involved. It is a problem which students of all ages can approach. It is a bit reminiscent of logic problems we tried when we were in elementary school: "Al, Betty, and Charles have three different kinds of pets and live in three different coloured houses"—and then you are given a bunch of seemingly unconnected statements and asked who owns the schnauzer.

Solution. The maximum score any student can have over the five events is $5 \times 5 = 25$ points. Since A scores 24 points in total, she must have scored 5 in four events and 4 in the other event. Since E scores 5 in event W, then A must score 4 in W and 5 in the rest of the events.

At this stage, it is a good idea to make a table:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B						
C						
D	4					
E		5	3			

So far it does not look good! However, there is an unusual approach—to look at the total scores. Note that the total of the five students' total scores must be $5(5 + 4 + 3 + 2 + 1) = 75$.

What is the minimum possible total score for E? Since E must get at least 1 in each of the three unspecified events, this minimum must be $1 + 5 + 3 + 1 + 1 = 11$. Now, could E have scored 12 or more points? If E scores 12, then D scores at least 13, C at least 14, and B at least 15, since there are no ties. But then the total of the five students' totals is at least $24 + 15 + 14 + 13 + 12 = 78$, which is impossible.

Therefore, E must have a total of 11 (hence, 1 in each of the three remaining events). Thus, E and A together score 35 points, leaving 40 points for B, C, and D, with each of these students scoring at least 12. But since none of the these totals can be equal, they must be 12, 13, and 15. (Try fiddling around with this to convince yourself!)

Remarkably, we now know the total scores for each student, without knowing all of the scores in the individual events. Thus, we now have:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B						15
C						13
D	4					12
E	1	5	3	1	1	11

At this stage, we should look back at the information we were given to see what we have not yet used. We still have not used the fact that C scores the same score in four different events. This score cannot be 1, 4, or 5, since we already have at least two of each of these scores in our table. If C scored 2 in four different events, then to get a total of 13, he must have scored 5 in the remaining event, which is impossible, since each event already has a score of 5 accounted for. Thus, C scores 3 in four different events (and 1 in the other event, since C's total must be 13). Looking at the entries already in the table, C must score 1 in event X. This gives us:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B						15
C	3	3	1	3	3	13
D	4					12
E	1	5	3	1	1	11

Now D scores 8 points in the last four events, and must score 1 or 2 in event W, and 2 or 4 in events X, Y, and Z. Since D cannot have only one odd score and an even total, then she must score 2 in event W, and thus she scores 2 in the rest of the events to get a total of 12. (A score of 4 would push her above this total score.) We can then quickly fill in B's scores by a process of elimination:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B	2	1	4	4	4	15
C	3	3	1	3	3	13
D	4	2	2	2	2	12
E	1	5	3	1	1	11

We have figured out all of the scores. These are the only possible scores because, in each step of our argument, there was only one possibility.

As this is the last Problem of the Month before summer, I would like to thank Shawn Godin for asking me to write this column. I hope that you have enjoyed the problems this year, and I look forward to continuing in the fall. Have a good summer!