MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier octobre 2005**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M194. *Proposé par Équipe de Mayhem.*

On suppose que $n-1$ et $n+1$ sont des premiers jumeaux, où $n \in \mathbb{N}$ et $n \geq 3$. Montrer que 1, 2, 3, $\dots$, n peuvent être arrangés dans une liste de telle sorte que la somme de deux éléments consécutifs quelconques est un nombre premier. (Par exemple, si $n=6$, un tel arrangement est 6, 5, 2, 1, 4, 3.)

M195. *Proposé par J. Walter Lynch, Athens, GA, USA.*

On divise un fil de longueur 1 en trois morceaux qu'on déforme pour en faire un carré, un cercle et un triangle équilatéral ayant les trois la même aire. Trouver la longueur de chacun des morceaux de fil.

M196. *Proposé par Équipe de Mayhem.*

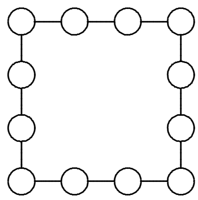
On cherche à former des comités à partir d'un groupe de personnes. Montrer que le nombre de comités possibles comportant un nombre impair de membres est exactement le même que le nombre de comités possibles comportant un nombre pair de membres.

M197. *Proposé par Neven Jurič, Zagreb, Croatie.*

Douze bateaux, occupant chacun trois cases horizontales ou verticales dans une grille 10×10, sont désignés par les lettres de A à L comme indiqué dans la figure. Chaque bateau contient un certain nombre de passagers. Les nombres qui figurent à droite de la dernière colonne et en-dessous de la dernière ligne indiquent le total des passagers de tous les bateaux touchés par la ligne ou la colonne correspondante. Par exemple, les deux bateaux B et L de la dernière colonne contiennent en tout 9 passagers. Quel est le nombre de passagers de chacun des douze bateaux, sachant que deux d'entre eux n'en contiennent aucun et que les dix restants en ont 1, 2, 3, 4, 5, 6, 7, 8, 9 et 10 ?

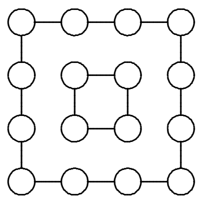
A	A	A							B	
					C	C	C		B	
	E	D	D	D					B	17
	E							J		
	E					I		J		15
						I		J		
F	F	F				I	K			19
		G					K		L	
		G	H	H	H		K		L	
		G							L	6
8		21			6				9	

M198. *Proposé par Équipe de Mayhem.*



On doit placer chacun des entiers de 1 à 12 dans un des cercles de la figure de telle sorte que, sur chaque côté de la figure, la somme des entiers soit 25. Trouver la somme des quatre entiers placés dans les coins.

M199. *Proposé par Équipe de Mayhem.*



Ce problème est une modification du problème précédent. Ici, on demande d'utiliser chacun des entiers de 1 à 16 de telle sorte que les quatre entiers des coins extérieurs aient la même somme que les quatre entiers formant la figure intérieure. Quelle est la somme maximale qu'on puisse obtenir, si c'est possible ?

M200. *Proposé par Équipe de Mayhem.*

Dans un carré d'aire unitaire on dessine deux droites perpendiculaires passant par son centre, le divisant ainsi en 4 morceaux. Quelle est l'aire maximale possible pour l'un quelconque d'entre eux ? Justifiez votre réponse.

M194. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Suppose $n - 1$ and $n + 1$ are twin primes where $n \in \mathbb{N}$ with $n \geq 3$. Show that 1, 2, 3, $\ldots$, n can be arranged in a row so that the sum of any two consecutive numbers is prime. (For example, when $n = 6$, one such arrangement is 6, 5, 2, 1, 4, 3.)

M195. *Proposed by J. Walter Lynch, Athens, GA, USA.*

A wire of unit length is divided into three pieces, which are used to construct a square, a circle, and an equilateral triangle such that each of them has the same area. Find the length of each of the three pieces of wire.

M196. *Proposed by the Mayhem Staff.*

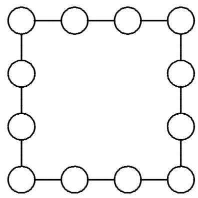
Committees are to be formed from a group of people. Show that the number of possible committees that can be formed with an odd number of members is exactly the same as the number of possible committees that can be formed with an even number of members.

M197. *Proposed by Neven Jurič, Zagreb, Croatia.*

There are twelve ships situated on a 10×10 grid. The ships are denoted by the letters A through L, and each ship consists of three cells of the grid in either a horizontal or a vertical line, as shown in the diagram. Each ship contains a certain number of passengers. There are also some numbers in the last row and the last column of the diagram. These numbers represent the total number of passengers on all the ships intersected by that row or column. For example, the two ships B and L in the last (right-most) column together contain 9 passengers. How many passengers does each of the twelve ships contain, if there are no passengers on two of the ships and the remaining ten ships contain 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 passengers?

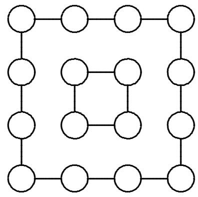
A	A	A							B	
					C	C	C		B	
	E	D	D	D					B	17
	E							J		
	E					I		J		15
						I		J		
F	F	F				I	K			19
		G					K		L	
		G	H	H	H		K		L	
		G							L	6
8		21			6				9	

M198. *Proposed by the Mayhem Staff.*



Each of the integers from 1 to 12 is to be placed in one of the circles in the figure so that the sum of the integers along each side of the figure is 25. Determine the sum of the four integers placed in the corners.

M199. *Proposed by the Mayhem Staff.*



This is a modification of the previous problem. In this case, the requirement is to use all the integers from 1 to 16 once each so that the integers along each of the four outer edges of the large figure and the four integers that make up the inner figure have identical sums. What is the largest sum, if any, that can be obtained?

M200. *Proposed by the Mayhem Staff.*

Two perpendicular lines are drawn through the centre of a square with area 1 square unit, cutting the square into 4 pieces. What is the largest possible area for any of the pieces? Justify your answer.

Problem of the Month

Ian VanderBurgh, University of Waterloo

Problem (2004 United Kingdom Mathematics Trust Junior Math Olympiad) At a summer camp, five students, called A, B, C, D, and E, each take part in five events, called V, W, X, Y, and Z. In each competition scores of 5, 4, 3, 2, and 1 are awarded for 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, and 5^{th}, respectively. There are no ties. Student A scores a total of 24 points, student C scores the same in each of four events, student D scores 4 in competition V, and student E scores 5 in W and 3 in X. Surprisingly, their overall positions are in alphabetical order. There are no ties in the final standings. Show that this information is enough to find all the scores, and that there is only one solution.

Trying to solve this problem is a good exercise in logic, as well as numerical manipulation. There is no high-level mathematics involved. It is a problem which students of all ages can approach. It is a bit reminiscent of logic problems we tried when we were in elementary school: "Al, Betty, and Charles have three different kinds of pets and live in three different coloured houses"—and then you are given a bunch of seemingly unconnected statements and asked who owns the schnauzer.

Solution. The maximum score any student can have over the five events is $5 \times 5 = 25$ points. Since A scores 24 points in total, she must have scored 5 in four events and 4 in the other event. Since E scores 5 in event W, then A must score 4 in W and 5 in the rest of the events.

At this stage, it is a good idea to make a table:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B						
C						
D	4					
E		5	3			

So far it does not look good! However, there is an unusual approach—to look at the total scores. Note that the total of the five students' total scores must be $5(5+4+3+2+1) = 75$.

What is the minimum possible total score for E? Since E must get at least 1 in each of the three unspecified events, this minimum must be $1+5+3+1+1 = 11$. Now, could E have scored 12 or more points? If E scores 12, then D scores at least 13, C at least 14, and B at least 15, since there are no ties. But then the total of the five students' totals is at least $24+15+14+13+12 = 78$, which is impossible.

Therefore, E must have a total of 11 (hence, 1 in each of the three remaining events). Thus, E and A together score 35 points, leaving 40 points for B, C, and D, with each of these students scoring at least 12. But since none of the these totals can be equal, they must be 12, 13, and 15. (Try fiddling around with this to convince yourself!)

Remarkably, we now know the total scores for each student, without knowing all of the scores in the individual events. Thus, we now have:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B						15
C						13
D	4					12
E	1	5	3	1	1	11

At this stage, we should look back at the information we were given to see what we have not yet used. We still have not used the fact that C scores the same score in four different events. This score cannot be 1, 4, or 5, since we already have at least two of each of these scores in our table. If C scored 2 in four different events, then to get a total of 13, he must have scored 5 in the remaining event, which is impossible, since each event already has a score of 5 accounted for. Thus, C scores 3 in four different events (and 1 in the other event, since C's total must be 13). Looking at the entries already in the table, C must score 1 in event X. This gives us:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B						15
C	3	3	1	3	3	13
D	4					12
E	1	5	3	1	1	11

Now D scores 8 points in the last four events, and must score 1 or 2 in event W, and 2 or 4 in events X, Y, and Z. Since D cannot have only one odd score and an even total, then she must score 2 in event W, and thus she scores 2 in the rest of the events to get a total of 12. (A score of 4 would push her above this total score.) We can then quickly fill in B's scores by a process of elimination:

	V	W	X	Y	Z	Total
A	5	4	5	5	5	24
B	2	1	4	4	4	15
C	3	3	1	3	3	13
D	4	2	2	2	2	12
E	1	5	3	1	1	11

We have figured out all of the scores. These are the only possible scores because, in each step of our argument, there was only one possibility.

As this is the last Problem of the Month before summer, I would like to thank Shawn Godin for asking me to write this column. I hope that you have enjoyed the problems this year, and I look forward to continuing in the fall. Have a good summer!

Pólya's Paragon

Fun With Numbers (Part 4)

Shawn Godin

Last issue we introduced the idea of modular arithmetic, and we looked at using digital sums to check calculations. The digital sum of a number is actually a single digit congruent to the number modulo 9. For example, from last issue, $43\,658\,912 \equiv 38 \equiv 11 \equiv 2 \pmod 9$. When we check a calculation using digital sums, we are really checking whether our answer is correct modulo 9.

How can we show this congruence between a number and its digital sum modulo 9? It becomes rather simple when we remember that an n–digit number $d_{n-1}d_{n-2}\ldots d_2d_1d_0$ is really the number

$$d_{n-1} \times 10^{n-1} + d_{n-2} \times 10^{n-2} + \cdots + d_2 \times 10^2 + d_1 \times 10 + d_0 .$$

Now we simply have to notice that $10 \equiv 1 \pmod{9}$, which implies that $10^k \equiv 1 \pmod{9}$ for any k. Hence,

$$\begin{aligned} & d_{n-1} \times 10^{n-1} + d_{n-2} \times 10^{n-2} + \cdots + d_2 \times 10^2 + d_1 \times 10 + d_0 \\ & \quad \equiv \; d_{n-1} \times 1^{n-1} + d_{n-2} \times 1^{n-2} + \cdots + d_2 \times 1^2 + d_1 \times 1 + d_0 \\ & \quad \equiv \; d_{n-1} + d_{n-2} + \cdots + d_2 + d_1 + d_0 \pmod{9} . \end{aligned}$$

The process of calculating the digital sum of a number has often been called "*casting out nines*", because when you calculate the digital sum you can ignore multiples of 9. For example, if you were to look at $43\,658\,912$, your first step would be to get rid of the 9s to get $43\,658\,\not{9}12$. Then, since $4 + 5 = 9$, $3 + 6 = 9$, and $8 + 1 = 9$, we get

$$\not{4}\not{3}\;\not{6}\not{5}\not{8}\;\not{9}\not{1}2 .$$

Therefore, the digital sum is 2, and we must have $43\,658\,912 \equiv 2 \pmod{9}$. Use this trick to amaze your parents and impress your peers!

This can be turned into a divisibility test, since $9 \mid a$ if and only if $a \equiv 0 \pmod{9}$ (why?).

Divisibility by 9: A number is divisible by 9 if and only if its digital sum is divisible by nine.

We can extend this idea a little further. By noting that $10^2 = 100$, we can convert a number from base ten to base hundred quite easily by looking at blocks of two digits starting from the right. For example,

$$\begin{aligned} 43\,658\,912 &= 4 \times 10^7 + 3 \times 10^6 + 6 \times 10^5 + 5 \times 10^4 \\ &\qquad + 8 \times 10^3 + 9 \times 10^2 + 1 \times 10 + 2 \\ &= 43 \times 100^3 + 65 \times 100^2 + 89 \times 100 + 12 \\ &= 43{,}65{,}89{,}12_{100} , \end{aligned}$$

where the commas separate the "digits" and the subscript 100 means we are working in base hundred instead of base ten. Then, since $100 \equiv 1 \pmod{99}$, we can see that the digital sum will be equivalent to the number modulo 99. That is, $43 + 65 + 89 + 12 = 209$ and $2 + 09 = 11$. This implies that $43\,658\,912 \equiv 11 \pmod{99}$.

So what? you say. How often do you want to divide something by 99? The real bonus here is that $43\,658\,912 \equiv 11 \pmod{99}$, which tells us that $43\,658\,912 = 99k + 11$ for some integer k. Since we know that $99 = 9 \times 11$, we see that $99 \equiv 0 \pmod{9}$ and $99 \equiv 0 \pmod{11}$, which gives us

$$43\,658\,912 \equiv 11 ,$$

using either $\bmod 9$ or $\bmod 11$. This means, in effect, that we have come up with one test that tests for divisibility by *both* 9 and 11.

We can use the structure of our number system to come up with other divisibility rules. The next two should be well known:

Divisibility by 2: A number is divisible by 2 if and only if its last digit is 0, 2, 4, 6, or 8.

Divisibility by 5: A number is divisible by 5 if and only if its last digit is either 0 or 5.

We can justify each of these two rules by noting that $10 = 2 \times 5$. Thus, $10 \equiv 0 \pmod{2}$ and $10 \equiv 0 \pmod{5}$. As a result, we have

$$d_{n-1}d_{n-2}\ldots d_2d_1d_0 \;\equiv\; d_0\,,$$

using either $\bmod 2$ or $\bmod 5$. We can extend this idea by noting that $4 = 2^2$. Then $100 = 10^2 \equiv 0 \pmod{4}$. Similarly, for $25 = 5^2$ we get $100 = 10^2 \equiv 0 \pmod{25}$. Thus,

$$\begin{aligned} & d_{n-1}d_{n-2}\ldots d_2d_1d_0 &\equiv&\; d_1 \times 10 + d_0 \pmod{4} \\ \text{and}\quad & d_{n-1}d_{n-2}\ldots d_2d_1d_0 &\equiv&\; d_1 \times 10 + d_0 \pmod{25}\,. \end{aligned}$$

From this we get

Divisibility by 4: A number is divisible by 4 if and only if the two-digit number formed by its last two digits is divisible by 4.

Divisibility by 25: A number is divisible by 25 if and only if the two-digit number formed by its last two digits is divisible by 25 (that is, 00, 25, 50, or 75).

This should be enough to get you going. Here are some things to try for homework.

1. Modify the divisibility test for 9 to get another test for 3.
2. Modify the divisibility test for 9 to get another test for 11. *Hint:* Note that $10 \equiv -1 \pmod{11}$. (Why?)
3. Construct divisibility tests for 2^n and 5^n for any integer $n > 1$.
4. Develop divisibility tests for 101, 1001, 10001, $\ldots$ and 999, 9999, 99999, $\ldots$, and see what else comes out of it.

One last note: the divisibility rule for a composite number is just a combination of the rules for the prime powers which are factors of the number. For example, to test divisibility by 12, you would just test for divisibility by 3 and 2^2, since a number is divisible by 12 if and only if it is divisible by both 3 and 2^2.

Have a great summer! See you again in September!

Misère Games

Arthur Holshouser and Harold Reiter

Abstract.

The theory of last-player-winning counter-pickup games is well known. See [1] and [2]. The corresponding *misère* games in which the last player loses are less well understood. In this note, we define a special class of combinatorial games and find the winning strategies for all composite games with these special games as components. In the first section we recall the method of using Nim values of component games to solve a composite game. In the second section, we define *special* games and find winning strategies for misère games.

Section 1: Terminology

Definition. A finite impartial game G played under normal rules of play is called a *regular* game. This means that

1. two players alternate moving,
2. there is no infinite sequence of moves,
3. both players have the same moves available, and
4. the winner is the last player to make a move.

Such a game can be thought of as a directed acyclic graph. Each *vertex* of the graph corresponds to a position in the game and each *directed edge* corresponds to a move. The *followers* of a vertex are those positions joined to it by an outgoing edge. We will briefly say that G is a regular impartial game.

Nim Values. The minimum excluded value, denoted mex, of a finite set of non-negative integers is the least non-negative integer not in the set. For example, $\text{mex}\{1, 2, 4, 0\} = 3$, $\text{mex}\{2, 4, 5\} = 0$, $\text{mex}\{\} = 0$. The *Nim value* of a position, denoted by $g(n)$, is the mex of the Nim values of its followers. A position with no followers (that is, a *terminal* position) has Nim value 0. It is easy to see that the winning strategy is to move to a position with Nim value 0, for then the opponent either has no move at all and loses immediately, or must move to a position with Nim value greater than 0 and hence must eventually lose.

Composite Games. *Composite games*, denoted $G = G_1 \oplus G_2 \oplus \cdots \oplus G_k$, are games that have several components. Two players alternate moves. Each player on his turn selects a component game G_i in which a legal move can be made and makes a legal move in that game. The winner is the last player to move. We define the Nim value of the composite game as the Nim sum (denoted by $\oplus$) of the Nim values of each of the component games. The Nim sum is obtained by writing the integers in binary and adding modulo 2 without carrying. For example, $6 \oplus 3 = 110_{(2)} \oplus 11_{(2)} = 101_{(2)} = 5_{(10)}$, since, by considering the digits in the summation from left to right, we get $1 \equiv 1 \pmod 2$, $1 + 1 \equiv 0 \pmod 2$, and $0 + 1 \equiv 1 \pmod 2$.

Strategy. The *balanced* positions are those positions whose Nim values are 0. The *unbalanced* positions are those positions whose Nim values are not zero.

If a position is balanced, it will always become unbalanced after the moving player moves. This follows from the definition of mex since if a player moves from n_i to m_i in G_i, then $g(n_i) \neq g(m_i)$.

Also, if a position is unbalanced, the moving player can always move to a balanced position. Such a winning move can always be selected from the component G_i that contributes the left-most 1 in the Nim sum of the component values. This follows from the definition of mex since if $g(n_i) \geq 1$ in game G_i, the moving player can move in G_i to a vertex m_i having any of the values $\{0, 1, 2, \ldots, g(n_i) - 1\}$. In particular, the moving player can move to a position m_i whose value is the sum of the Nim values of the other components. Of course, all terminal positions have a Nim value of $0 \oplus 0 \oplus \cdots \oplus 0 = 0$, which is balanced.

Section 2

***Misère* version of a game**. The misère version of a regular impartial game G_i is played by the same rules as G_i except the loser is the player who makes the last move.

The misère version of a composite game $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ is played by the same rules as $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ except the loser is the last player to move.

Special Games Suppose G is a regular impartial game. We say that G_i is *special* if, for each position n in G, when $g(n) = 0$, we have either (i) n is a terminal position or (ii) there exists a follower m of n such that $g(m) = 1$.

Problem 1. Suppose $G_1, G_2, \ldots, G_k$ are special, regular impartial games. Find a strategy for playing the misère version of $G_1 \oplus G_2 \oplus \cdots \oplus G_k$.

Solution. Let $(n_1, n_2, \ldots, n_k)$ denote an arbitrary position in the composite game $G_1 \oplus G_2 \oplus \cdots \oplus G_k$. We will first define the balanced positions.

A. If each $g(n_i) \in \{0, 1\}$, then $(n_1, n_2, \ldots, n_k)$ is balanced if and only if $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 1$.

B. If at least one $g(n_i) \notin \{0, 1\}$, then $(n_1, n_2, \ldots, n_k)$ is balanced if and only if $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 0$.

Let B, U denote the balanced and unbalanced positions respectively.

We note that all terminal positions, which we denote 0, are unbalanced. We will prove the following which we have illustrated in Figure 1.

(1) If $(n_1, n_2, \ldots, n_k)$ is balanced, then all moves must be to an unbalanced position.

(2) If $(n_1, n_2, \ldots, n_k)$ is unbalanced and non-terminal, then there exists a move to a balanced position.

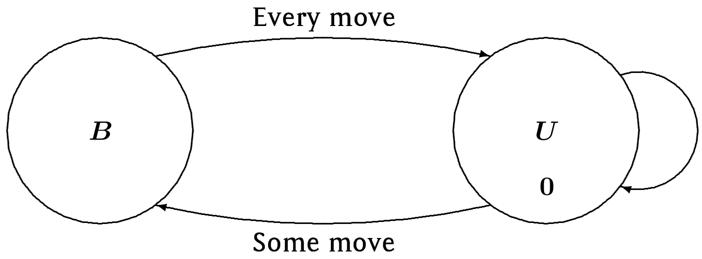


Figure 1

From (1) and (2) it follows that if $(n_1, n_2, \ldots, n_k)$ is the initial position in the game, then

(a) if $(n_1, n_2, \ldots, n_k)$ is balanced, the first player to move will lose if the opposing player plays perfectly.

(b) if $(n_1, n_2, \ldots, n_k)$ is unbalanced, then the first player to move will win with perfect play.

We now prove (1) and (2).

Proof of (1). For the balanced position $(n_1, n_2, \ldots, n_k)$, we consider two cases.

Case (a). $g(n_i) \in \{0, 1\}$ for all i.

Since the position is balanced, we have $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 1$. Without loss of generality, we may assume the player to move chooses to make a move in game G_1, which must be non-terminal, of course.

If $g(n_1) = 0$, then, by the definition of mex, the player to move must move to m_1 with $g(m_1) = 1$ or $g(m_1) \geq 2$. In either case, the new position, $(m_1, n_2, n_3, \ldots, n_k)$, is unbalanced.

If $g(n_1) = 1$, then, by the definition of mex, the player to move must move to m_1 with $g(m_1) = 0$ or $g(m) \geq 2$. In either case, the new position, $(m_1, n_2, n_3, \ldots, n_k)$, is unbalanced.

Case (b). $g(n_i) \notin \{0, 1\}$ for some i.

Then $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 0$, which implies there must also be a $j \neq i$ such that $g(n_j) \notin \{0, 1\}$. Now after the next move, there must still exist a game G_j such that $g(n_j) \notin \{0, 1\}$. By the definition of mex, after this next move it will be impossible for $g(\overline{n_1}) \oplus g(\overline{n_2}) \oplus \cdots \oplus g(\overline{n_k}) = 0$ where $(\overline{n_1}, \overline{n_2}, \ldots, \overline{n_k})$ is the new position. Therefore, $(\overline{n_1}, \overline{n_2}, \ldots, \overline{n_k})$ is unbalanced.

Proof of (2). For the unbalanced non-terminal position $(n_1, n_2, \ldots, n_k)$, we consider the two cases.

Case (a). $g(n_i) \notin \{0, 1\}$ for some i.

(i) Only one $g(n_i) \notin \{0, 1\}$. Since $g(n_i) \geq 2$, by the definition of mex, the player to move can move to an m_i such that $g(m_i) = 0$ and move to an $\overline{m_i}$ such that $g(\overline{m_i}) = 1$. This easily implies that he can move to a balanced position.

(ii) Two or more $g(n_i) \notin \{0, 1\}$. By the definition of mex, the player to move (as in Bouton's Nim) moves to a position $(\overline{n_1}, \overline{n_2}, \ldots, \overline{n_k})$ such that $g(\overline{n_1}) \oplus g(\overline{n_2}) \oplus \cdots \oplus g(\overline{n_k}) = 0$, which is a balanced position.

Case (b). $g(n_i) \in \{0, 1\}$ for all i.

Since the position $(n_1, n_2, \ldots, n_k)$ is unbalanced, it follows that $g(n_1) \oplus g(n_2) \oplus \cdots \oplus g(n_k) = 0$. Now, since $(n_1, n_2, \ldots, n_k)$ is non-terminal, let n_i be a non-terminal vertex in a game G_i. If $g(n_i) = 1$, by the definition of mex, the player to move can move to m_i with $g(m_i) = 0$, which balances the game. If $g(n_i) = 0$, by the definition of a special game, the player to move can move to m_i with $g(m_i) = 1$, which again balances the game. ■

References

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