SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2608★. [2001 : 49; 2002 : 181–184] *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that x, y, $z \geq 0$ and $x^2+y^2+z^2=1$. Prove or disprove that

(a) $1 \;\leq\; \dfrac{x}{1-yz}+\dfrac{y}{1-zx}+\dfrac{z}{1-xy} \;\leq\; \dfrac{3\sqrt{3}}{2}$;

(b) $1 \;\leq\; \dfrac{x}{1+yz}+\dfrac{y}{1+zx}+\dfrac{z}{1+xy} \;\leq\; \sqrt{2}$.

Solution by B.J. Venkatachala, Indian Institute of Science, Bangalore, India.

For the inequality on the right in part (b), the only solution given in [2002 : 183–184] used the method of Lagrange Multipliers. We prove it using only elementary algebra.

Without loss of generality, we assume that $x \leq z$ and $y \leq z$. Then

$$\sum_{\text{cyclic}} \frac{x}{1+yz} \;\leq\; \frac{x+y+z}{1+xy},$$

and it is sufficient to prove that

$$x+y+z \;\leq\; \sqrt{2}(1+xy),$$

for $0 \leq x \leq z$, $0 \leq y \leq z$, and $x^2+y^2+z^2=1$. That is,

$$x+y+\sqrt{1-x^2-y^2} \;\leq\; \sqrt{2}(1+xy).$$

This is further equivalent to

$$1-x^2-y^2 \;\leq\; \left(\sqrt{2}(1+xy)-x-y\right)^2.$$

Setting $\alpha = x+y$ and $\beta = xy$, this inequality becomes

$$1-\alpha^2+2\beta \;\leq\; \left(\sqrt{2}(1+\beta)-\alpha\right)^2.$$

This simplifies to

$$\left(\sqrt{2}\alpha-\beta-1\right)^2+\beta^2 \;\geq\; 0.$$

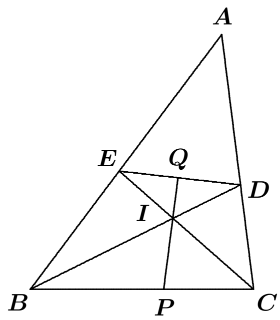
Since the left side is a sum of squares, the result follows.

We observe that equality occurs only when $\beta = 0$ and $\sqrt{2}\alpha - \beta - 1 = 0$; that is, $xy = 0$ and $x + y = 1/\sqrt{2}$. We conclude that we get equality only when $(x,y,z) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ or $(x,y,z) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. By removing the artificially imposed restrictions that $x \le z$ and $y \le z$, we obtain a third (symmetric) possibility, namely $(x,y,z) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

2939. [2004 : 229, 232] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $\triangle ABC$ has incentre I and that BI, CI meet AC, AB at D, E, respectively. Suppose further that the bisector of $\angle BIC$ meets BC and DE at P and Q, respectively, and that $PI = 2QI$. Prove that $\angle BAC = 60^\circ$.

I. *Solution by Michel Bataille, Rouen, France.*



With the usual notations, the area of $\triangle ABC$ is $\frac{1}{2}bc\sin A$. This area is also $\frac{1}{2}w_a c\sin\left(\frac{1}{2}A\right) + \frac{1}{2}w_a b\sin\left(\frac{1}{2}A\right)$, where w_a is the length of the internal bisector of $\angle BAC$. It follows that $\left(\frac{1}{b} + \frac{1}{c}\right) w_a = 2\cos\left(\frac{1}{2}A\right)$.

Applying this result to $\triangle BIC$ and $\triangle DIE$ in place of $\triangle ABC$, we obtain

$$\begin{aligned}\left(\frac{1}{BI} + \frac{1}{CI}\right) IP &= 2\cos\left(\tfrac{1}{2}\angle BIC\right)\\ &= 2\cos\left(\tfrac{1}{2}\angle DIE\right) = \left(\frac{1}{DI} + \frac{1}{EI}\right) IQ\,.\end{aligned}$$

By hypothesis, we have $PI = 2QI$. Hence,

$$2\left(\frac{1}{BI} + \frac{1}{CI}\right) = \left(\frac{1}{DI} + \frac{1}{EI}\right)\,. \qquad (1)$$

Letting r denote the inradius, we have

$$\frac{1}{BI} + \frac{1}{CI} = \frac{\sin(\frac{1}{2}B) + \sin(\frac{1}{2}C)}{r}$$

and

$$\begin{aligned}\frac{1}{DI} + \frac{1}{EI} &= \frac{\sin(\angle ADI) + \sin(\angle AEI)}{r}\\ &= \frac{\sin(A + \frac{1}{2}B) + \sin(A + \frac{1}{2}C)}{r}\,.\end{aligned}$$

Therefore, equation (1) may be written successively as

$$\begin{aligned}2\left(\sin(\tfrac{1}{2}B) + \sin(\tfrac{1}{2}C)\right) &= \sin(A + \tfrac{1}{2}B) + \sin(A + \tfrac{1}{2}C)\,,\\ 2\sin\left(\frac{B+C}{4}\right)\cos\left(\frac{C-B}{4}\right) &= \sin\left(A + \frac{B+C}{4}\right)\cos\left(\frac{C-B}{4}\right)\,.\end{aligned}$$

Note that $\cos\left(\frac{C-B}{4}\right) \neq 0$, since $\frac{C-B}{4} \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$. Hence,

$$\begin{aligned} 2\sin\left(\frac{\pi-A}{4}\right) &= \sin\left(\frac{\pi+3A}{4}\right), \\ \sin\left(\frac{\pi-A}{4}\right) &= \sin\left(\frac{\pi-3A}{4}\right) - \sin\left(\frac{\pi-A}{4}\right) \\ &= 2\sin\left(\frac{A}{2}\right)\sin\left(\frac{\pi-A}{4}\right). \end{aligned}$$

Note that $\sin\left(\frac{\pi-A}{4}\right) \neq 0$, since $\frac{\pi-A}{4} \in \left(0, \frac{\pi}{4}\right)$. Hence, $1 = 2\sin\left(\frac{1}{2}A\right)$, and the result follows immediately.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

As usual, let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$. Then it is well known that $BD = \frac{2\sqrt{acs(s-b)}}{a+c}$. Also, $\frac{BI}{ID} = \frac{BC}{DC} = \frac{a+c}{b}$. Hence,

$$BI = \sqrt{\frac{ac(s-b)}{s}} \qquad \text{and} \qquad ID = \frac{b}{a+c}\sqrt{\frac{ac(s-b)}{s}},$$

Similarly,

$$CI = \sqrt{\frac{ab(s-c)}{s}} \qquad \text{and} \qquad IE = \frac{c}{a+b}\sqrt{\frac{ab(s-c)}{s}}.$$

Now, let $\theta = \angle BIP$. Then

$$\tfrac{1}{2}BI \cdot CI \sin(2\theta) = [BIC] = [BIP] + [PIC] = \tfrac{1}{2}PI(BI+CI)\sin\theta.$$

Hence, $PI = \frac{2BI \cdot CI \cos\theta}{BI+CI}$. Likewise, $IQ = \frac{2ID \cdot IE \cos\theta}{ID+IE}$. Substituting the above expressions for BI, CI, ID and IE, and setting $PI = 2IQ$, we get

$$\frac{1}{\sqrt{c(s-b)} + \sqrt{b(s-c)}} = \frac{2\sqrt{bc}}{(a+b)\sqrt{b(s-b)} + (a+c)\sqrt{c(s-c)}},$$

which can be transformed into

$$\left(2\sqrt{(s-b)(s-c)} - \sqrt{bc}\right)\left(\sqrt{b(s-c)} + \sqrt{c(s-b)}\right) = 0.$$

Hence, $4(s-b)(s-c) = bc$. Then $\cos A = 1 - \frac{2(s-b)(s-c)}{bc} = \frac{1}{2}$, so that $A = 60^\circ$.

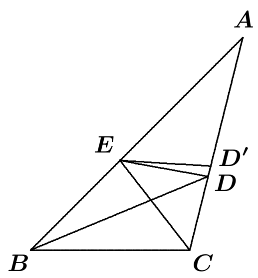
Comment: As a consequence, $\triangle DEP$ is equilateral with I as its centroid.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2940. [2004 : 229, 232] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the bisectors of $\angle ABC$ and $\angle ACB$ meet AC and AB at D and E, respectively, and $\angle ADE - \angle AED = 60^\circ$. Prove that $\angle ACB = 120^\circ$.

Composite of similar solutions by Peter Y. Woo, Biola University, La Mirada, CA, USA; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.



We will prove the stronger result that $\angle ADE - \angle AED = 60^\circ$ if and only if $\angle ACB = 120^\circ$.

Since BD bisects $\angle B$, we get $\frac{DC}{DA} = \frac{BC}{BA}$. Using the Sine Law in $\triangle AEC$ and then in $\triangle ABC$, we get

$$\begin{aligned}\frac{EC}{AE} &= \frac{\sin\angle EAC}{\sin\angle ECA} = \frac{\sin\angle A}{\sin\left(\frac{1}{2}\angle C\right)} = 2\cos\left(\tfrac{1}{2}\angle C\right)\frac{\sin\angle A}{\sin\angle C} \\ &= 2\cos\left(\tfrac{1}{2}\angle C\right)\frac{BC}{BA} = 2\cos\left(\tfrac{1}{2}\angle C\right)\frac{DC}{DA}.\end{aligned}$$

Let the angle bisector of $\angle AEC$ meet AC at D'. Then

$$\frac{D'C}{D'A} = \frac{EC}{AE} = 2\cos\left(\tfrac{1}{2}\angle C\right)\frac{DC}{DA}.$$

Suppose that $\angle C < 120^\circ$. Then $2\cos\left(\frac{1}{2}\angle C\right) > 1$, and thus we have $\frac{D'C}{D'A} > \frac{DC}{DA}$. It follows that the points C, D, D', and A lie in that order on AC. (This is the case shown in the diagram.) Thus,

$$\angle AED > \angle AED' = \angle CED' > \angle CED.$$

Hence,

$$\angle ADE - \angle AED < \angle ADE - \angle CED = \angle DCE = \tfrac{1}{2}\angle C < 60^\circ.$$

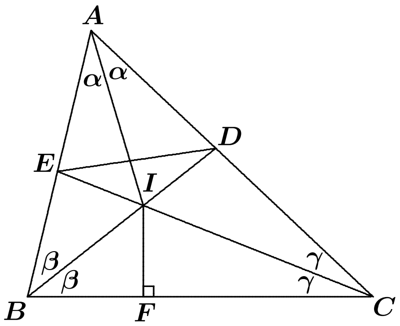
If $\angle C > 120^\circ$, then all the inequalities in the paragraph above are reversed, giving $\angle ADE - \angle AED > 60^\circ$. Finally, if $\angle C = 120^\circ$, then $D' = D$, and we find that $\angle ADE - \angle AED = 60^\circ$. The result follows.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; D.J. SMEENK, Zaltbommel, the Netherlands; IAN VANDERBURGH, University of Waterloo, Waterloo, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incomplete solution.

2941. [2004 : 229, 232] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the bisectors of $\angle ABC$ and $\angle ACB$ meet AC and AB at D and E, respectively. Let I be the intersection of BD and CE, and let F be the foot of the perpendicular from I to BC. Prove that if $\angle ADE = \angle BIF$, then $\angle AED = \angle CIF$.

Composite of almost identical solutions by John G. Heuver, Grande Prairie, AB; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.



Let $\alpha = \angle A/2$, $\beta = \angle B/2$, and $\gamma = \angle C/2$. Then $\alpha + \beta + \gamma = 90^\circ$. Note that

$$\angle ADE \;=\; \angle CED + \angle DCE \;=\; \angle CED + \gamma\,.$$

Also, using the hypothesis that $\angle ADE = \angle BIF$, we have

$$\angle ADE \;=\; \angle BIF \;=\; 90^\circ - \beta \;=\; \alpha + \gamma\,.$$

It follows that $\angle CED = \alpha$. Thus, $\angle IED = \angle IAD$. This implies that the points A, E, I, and D are concyclic. Then

$$\angle AED \;=\; \angle AID \;=\; \angle BAI + \angle ABI \;=\; \alpha + \beta \;=\; 90^\circ - \gamma \;=\; \angle CIF\,.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; D.J. SMEENK, Zaltbommel, the Netherlands; IAN VANDERBURGH, University of Waterloo, Waterloo, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Comăneşti, Romania. Most solvers used trigonometry, the Law of Sines in particular.

Other consequences that may be deduced here are that $DI = EI$ and $\angle A = 60^\circ$.

2942. [2004 : 229, 232] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given $\triangle ABC$ with $\angle ABC = 2\angle ACB$, suppose that D is a point on the ray CB such that $\angle ADC = \frac{1}{2}\angle BAC$. Prove that

$$\frac{1}{CD} \;=\; \frac{1}{AB} - \frac{1}{AC}\,.$$

Solution by Richard B. Eden, Ateneo de Manila University, The Philippines.

Let $\angle ACB = 2\theta$. Then $\angle ABC = 4\theta$, $\angle BAC = 180^\circ - 6\theta$, and $\angle ADC = 90^\circ - 3\theta$. Now, $\angle DAB = 4\theta - (90^\circ - 3\theta) = 7\theta - 90^\circ$, so that $\angle DAC = (7\theta - 90^\circ) + (180^\circ - 6\theta) = \theta + 90^\circ$. Using the Sine Law for triangles DAC and BAC, we obtain

$$\begin{aligned}\frac{1}{CD} + \frac{1}{AC} &= \frac{\sin(90^\circ - 3\theta)}{AC\sin(90^\circ + \theta)} + \frac{1}{AC} \\ &= \frac{1}{AC}\left(\frac{\cos 3\theta}{\cos\theta} + 1\right) \\ &= \frac{\sin 2\theta}{AB\sin 4\theta}\left(\frac{\cos 3\theta + \cos\theta}{\cos\theta}\right) \\ &= \frac{1}{2AB\cos 2\theta}\cdot\frac{2\cos 2\theta\cos\theta}{\cos\theta} = \frac{1}{AB},\end{aligned}$$

which completes the proof.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; IAN VANDERBURGH, University of Waterloo, Waterloo, ON; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Note that the featured solution is valid only if $\theta \geq 90^\circ/7$. Then $4\theta \geq 90^\circ - 3\theta$, which implies that the point D is on the ray CB, beyond the point B (or it coincides with B). If $\theta < 90^\circ/7$, then the solution is similar; the only difference is that $\angle DAB$ is $90^\circ - 7\theta$, rather than $7\theta - 90^\circ$. In this case, $4\theta < 90^\circ - 3\theta < 90^\circ$, meaning that the point D lies on the segment CB. Villar Rubio was the only solver who considered both cases.

2943. [2004 : 230, 232] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given $\triangle ABC$, let D be the point on AB produced beyond B such that $BD = BC$, and let E be the point on AC produced beyond C such that $CE = BC$. Let P be the intersection of BE and CD, and suppose that $\frac{DP}{BE} + \frac{EP}{CD} = 2\sin\left(\frac{\angle BAC}{2}\right)$. Prove that $\angle BAC = 90^\circ$.

Solution by Michel Bataille, Rouen, France.

Let $AB = c$, $BC = a$ and $CA = b$. Since triangles DBC and ECB are isosceles, and since $\angle BCE = 180^\circ - C$ and $\angle CBD = 180^\circ - B$, we have $\angle CBP = \angle CEP = \frac{1}{2}C$ and $\angle BDP = \angle BCP = \frac{1}{2}B$. It follows that

$$\begin{aligned}\angle PCE &= 180^\circ - C - \tfrac{1}{2}B = A + \tfrac{1}{2}B, \\ \angle PBD &= 180^\circ - B - \tfrac{1}{2}C = A + \tfrac{1}{2}C, \\ \text{and}\quad \angle BPD &= \angle CPE = \tfrac{1}{2}(B + C) = 90^\circ - \tfrac{1}{2}A.\end{aligned}$$

Using the Sine Law for triangles BCE and CBD, we obtain

$$BE \;=\; 2a\cos\left(\tfrac{1}{2}C\right) \quad\text{and}\quad CD \;=\; 2a\cos\left(\tfrac{1}{2}B\right)\,.$$

Using the Sine Law for triangles DBP and ECP, we then obtain

$$DP \;=\; \frac{a}{\cos\left(\frac{1}{2}A\right)}\sin\left(A+\tfrac{1}{2}C\right) \quad\text{and}\quad EP \;=\; \frac{a}{\cos\left(\frac{1}{2}A\right)}\sin\left(A+\tfrac{1}{2}B\right)\,.$$

Thus, the hypothesis $\dfrac{DP}{BE}+\dfrac{EP}{CD} \;=\; 2\sin\left(\dfrac{\angle BAC}{2}\right)$ can be written as

$$\begin{aligned}\sin A &= \frac{1}{2}\left(\frac{\sin\left(A+\frac{1}{2}C\right)}{\cos\left(\frac{1}{2}C\right)}+\frac{\sin\left(A+\frac{1}{2}B\right)}{\cos\left(\frac{1}{2}B\right)}\right)\\ &= \sin A+\frac{1}{2}\cos A\left(\frac{\sin\left(\frac{1}{2}C\right)\cos\left(\frac{1}{2}B\right)+\sin\left(\frac{1}{2}B\right)\cos\left(\frac{1}{2}C\right)}{\cos\left(\frac{1}{2}B\right)\cos\left(\frac{1}{2}C\right)}\right)\,,\\ &= \sin A+\frac{1}{2}\cos A\left(\frac{\cos\left(\frac{1}{2}A\right)}{\cos\left(\frac{1}{2}B\right)\cos\left(\frac{1}{2}C\right)}\right)\,.\end{aligned}$$

It follows that $\cos A = 0$ and, therefore, $\angle BAC = A = 90^\circ$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2944. [2004 : 230, 232] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Given an ellipse with foci F_1 and F_2, minor vertices V_1' and V_2', a line ℓ, and a point P not on ℓ. Construct, with straightedge alone, the line through P which is

(a) parallel to ℓ;

(b) perpendicular to ℓ.

The constructions are well known, if a circle with its centre is given instead of an ellipse and its foci (Poncelet–Steiner Construction Theorem).

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) We freely use the following two constructions (that use unmarked straightedge only).

1. Given two points A and B and a line parallel to AB, we can locate the mid-point of AB.
2. Given A and B and their mid-point, we can draw a line parallel to AB through any point P.

[*Editor's comment*. Constructions by straightedge alone are the subject of problems 2694–2696 [2001 : 535; 2002 : 553–557], 2740 [2002 : 245; 2003 : 246], and 2741 [2002 : 245; 2003 : 325–328]. Further references are provided with the first set on page 557. The above constructions 1 and 2 can be found in the solution to 2695 [2002 : 553–554] and to 2741 [2003 : 326, construction II(b)].]

The given line ℓ must intersect one of F_1V_1' or F_1V_2'. Without loss of generality, assume that ℓ intersects F_1V_1' at M_1 and $V_2'F_2$ at M_2. Since F_1V_2' is parallel to $V_1'F_2$, using (1) we can construct the mid-points N_1 and N_2 of F_1V_2' and $V_1'F_2$, respectively. Then N_1N_2 intersects ℓ at the mid-point M of M_1M_2. The construction is then completed by (2).

(b) By (a) we can construct line m through F_2 and parallel to ℓ. By solution II to 2741 cited above, we can construct line n through F_2 and perpendicular to m. By (a) we can construct the line through P and parallel to n.

Also solved by Peter Y. Woo, Biola University, La Mirada, CA, USA; and the proposer.

2945. [2004 : 230, 233] *Proposed by Michel Bataille, Rouen, France.*

Let G be the centroid of $\triangle A_1A_2A_3$. For $j = 1, 2, 3$, a circle is tangent to A_jA_{j+1} at T_j and to A_jA_{j+2} at U_j, so that G lies on the line segment T_jU_j (subscripts are taken modulo 3). Prove that

$$|GT_1| \cdot |GT_2| \cdot |GT_3| \;=\; |GU_1| \cdot |GU_2| \cdot |GU_3| \,.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

In fact, the theorem is true if G is any point inside $\triangle ABC$. Using the Law of Sines, we have, for each i,

$$GT_i \;=\; A_iT_i\frac{\sin\angle GA_iT_i}{\sin\angle A_iGT_i} \quad\text{and}\quad GU_i \;=\; A_iU_i\frac{\sin\angle GA_iU_i}{\sin\angle A_iGU_i}\,.$$

Noting that $A_iT_i = A_iU_i$ and $\angle A_iGT_i + \angle A_iGU_i = \pi$, we get

$$\frac{GT_i}{GU_i} \;=\; \frac{\sin\angle GA_iT_i}{\sin\angle GA_iU_i} \;=\; \frac{\sin\angle GA_iA_{i+1}}{\sin\angle GA_iA_{i+2}}\,.$$

Hence,

$$\begin{aligned}\prod_{i=1}^{3}\frac{GT_i}{GU_i} \;&=\; \prod_{i=1}^{3}\frac{\sin\angle GA_iA_{i+1}}{\sin\angle GA_iA_{i+2}} \\ &=\; \prod_{i=1}^{3}\frac{\sin\angle GA_iA_{i+1}}{\sin\angle GA_{i+1}A_i} \;=\; \prod_{i=1}^{3}\frac{GA_{i+1}}{GA_i} \;=\; 1\,.\end{aligned}$$

The result follows immediately.

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

2946. [2004 : 230, 233] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let x, y, z be positive real numbers satisfying $x^2+y^2+z^2=1$. Prove that

(a) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z)\geq 2\sqrt{3}$.

(b) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+(x+y+z)\geq 4\sqrt{3}$.

I. *Solution by Arkady Alt, San Jose, CA, USA.*

(a) Note first that, for any $u>0$, the inequality $\frac{1}{u}-u\geq\frac{4\sqrt{3}}{3}-2\sqrt{3}u^2$ is equivalent to each of the following:

$$\begin{aligned} 6\sqrt{3}u^3-3u^2-4\sqrt{3}u+3 &\geq 0\,, \\ 2(\sqrt{3}u)^3-(\sqrt{3}u)^2-4(\sqrt{3}u)+3 &\geq 0\,, \\ (\sqrt{3}u-1)^2(2\sqrt{3}u+3) &\geq 0\,. \end{aligned}$$

The last inequality is clearly true. Hence,

$$\begin{aligned} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z) &= \sum_{\text{cyclic}}\left(\frac{1}{x}-x\right) \\ &\geq \sum_{\text{cyclic}}\left(\frac{4\sqrt{3}}{3}-2\sqrt{3}x^2\right) \;=\; 2\sqrt{3}\,. \end{aligned}$$

(b) For any $u>0$, the inequality $\frac{1}{u}+u\geq\frac{5\sqrt{3}}{3}-\sqrt{3}u^2$ is equivalent to each of the following:

$$\begin{aligned} 3\sqrt{3}u^3+3u^2-5\sqrt{3}u+3 &\geq 0\,, \\ (\sqrt{3}u)^3+(\sqrt{3}u)^2-5(\sqrt{3}u)+3 &\geq 0\,, \\ (\sqrt{3}u-1)^2(\sqrt{3}u+3) &\geq 0\,. \end{aligned}$$

The last inequality is clearly true. Hence,

$$\begin{aligned} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+(x+y+z) &= \sum_{\text{cyclic}}\left(\frac{1}{x}+x\right) \\ &\geq \sum_{\text{cyclic}}\left(\frac{5\sqrt{3}}{3}-\sqrt{3}x^2\right) \;=\; 4\sqrt{3}\,. \end{aligned}$$

II. *Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

(a) By the Root–Mean–Square Inequality, we have

$$\frac{x+y+z}{3} \;\le\; \sqrt{\frac{x^2+y^2+z^2}{3}} \;=\; \frac{1}{\sqrt{3}}$$

and hence,

$$x+y+z \;\le\; \sqrt{3}\,. \tag{1}$$

By the AM–HM Inequality, we have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;\ge\; \frac{9}{x+y+z} \;\ge\; 3\sqrt{3}\,. \tag{2}$$

From (1) and (2), the claim follows.

(b) We apply the AM–GM Inequality twice:

$$\begin{aligned}
&\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+(x+y+z)\\
&\qquad\ge\; 4\left(\frac{x+y+z}{xyz}\right)^{1/4} \;=\; 4\left(\frac{(x+y+z)(x^2+y^2+z^2)}{xyz}\right)^{1/4}\\
&\qquad=\; 4\left(\frac{x^2}{yz}+\frac{y}{z}+\frac{z}{y}+\frac{y^2}{zx}+\frac{z}{x}+\frac{x}{z}+\frac{z^2}{xy}+\frac{x}{y}+\frac{y}{x}\right)^{1/4}\\
&\qquad\ge\; 4\left(9\sqrt[9]{\frac{x^4y^4z^4}{x^4y^4z^4}}\right)^{1/4} \;=\; 4\sqrt[4]{9} \;=\; 4\sqrt{3}\,.
\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; VEDULA N. MURTY, Dover, PA, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; IAN VANDERBURGH, University of Waterloo, Waterloo, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; ROGER ZARNOWSKI, Angela State University, San Angela, TX, U.S.A., YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer. About two thirds of the submitted solutions used one or more of the following: AM–GM Inequality, AM–HM Inequality, AM–RMS Inequality, Cauchy–Schwarz Inequality. The rest used calculus, convexity and Jensen's Inequality. One solver used the method of Lagrange's Multipliers.

Many solvers remarked that equality holds in either of the two inequalities if and only if $x=y=z=1/\sqrt{3}$.

Bencze obtained the generalization that if $x_k > 0$ *(for* $k = 1, 2, \ldots, n$*) such that* $\sum\limits_{k=1}^{n} x_k^2 = 1$, *then for all* $a, b > 0$,

$$a\left(\sum_{k=1}^{n}\frac{1}{x_k}\right) \pm b\left(\sum_{k=1}^{n} x_k\right) \;\ge\; (an \pm b)\sqrt{n}\,.$$

Chung and Janous gave a similar comment which is the special case when $a = b = 1$. Indeed, Janous generalized even further: if $\lambda \geq \mu > 0$ and if $x_k > 0$ (for $k = 1, 2, \ldots, n$) such that $\sum_{k=1}^{n} x_k^{\lambda} = 1$, then

$$\left(\sum_{k=1}^{n} \frac{1}{x_k^{\mu}}\right) \pm \left(\sum_{k=1}^{n} x_k^{\mu}\right) \;\geq\; n^{(\lambda+\mu)/\lambda} \pm n^{(\lambda-\mu)/\lambda}\,.$$

2947★. [2004 : 230, 233] *Proposed by Abbas Mehrabian, student, Tehran, Iran.*

The featured solution to problem 2149 [1996 : 171; 1997 : 306–308] is missing a step, which we remedy by means of the following problem. Let $A'B'C'D'$ be a quadrilateral with an inscribed circle centred at O. For any point P inside $A'B'C'D'$, define $ABCD$ to be the convex quadrilateral whose sides pass through the vertices of $A'B'C'D'$ and are perpendicular at the vertex to the line joining it to P. Prove that P is the intersection point of the diagonals AC and BD if and only if $P = O$.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Let us label the points so that A', B', C', D' lie on DA, AB, BC, CD, respectively.

Claim 1. $\angle A'OB' = \angle A'D'O + \angle OC'B'$.

Proof: We have [using the fact that the incentre lies on the angle bisectors],

$$\begin{aligned}
&\angle A'D'O + \angle OC'B' - \angle A'OB' \\
&\qquad = \angle A'D'O + \angle OC'B' + \angle OB'A' + \angle B'A'O - 180^\circ \\
&\qquad = \tfrac{1}{2}\left(\angle C'D'A' + \angle B'C'D' + \angle A'B'C' + \angle D'A'B'\right) - 180^\circ \\
&\qquad = 180^\circ - 180^\circ = 0^\circ\,.
\end{aligned}$$

■

Suppose now that $P = O$. Using Claim 1 along with the fact that $OD'DA'$ and $OB'BC'$ are cyclic, we have

$$\begin{aligned}
&\angle DOA' + \angle A'OB' + \angle B'OB \\
&\qquad = 90^\circ - \angle A'DO + \angle A'OB' + 90^\circ - \angle OBB' \\
&\qquad = \angle A'OB' - \angle A'D'O - \angle OC'B' + 180^\circ \\
&\qquad = \angle A'OB' - \angle A'D'O - \angle D'C'O + 180^\circ \;=\; 180^\circ\,.
\end{aligned}$$

Hence, B, O, and D are collinear. Similarly, A, O, and C are collinear. It follows that $O = P$ is the intersection of AC and BD.

Conversely, suppose that $P \neq O$. We may assume, without loss of generality, that P is in or on the triangle $C'D'O$.

Claim 2. $\angle A'PB' < \angle A'OB'$.

Proof: Consider the tangent to the circle $A'OB'$ at O. From Claim 1 we have $\angle A'OD' = \angle A'B'O + \angle OC'D' > \angle A'B'O$; it follows that the tangent goes through the interior of $\angle A'OD'$ and, similarly, through the interior of $\angle B'OC'$. Hence, circle $A'OB'$ intersects triangle $C'D'O$ only at the point O. Thus, P is outside circle $A'OB'$, and $\angle A'PB' < \angle A'OB'$ as claimed. ∎

Assuming, for the moment, that the quadrangles $C'PB'B$ and $D'PA'D$ are convex, we have

$$\begin{aligned}
&\angle DPA' + \angle A'PB' + \angle B'PB \\
&\qquad = 90^\circ - \angle A'DP + \angle A'PB' + 90^\circ - \angle PBB' \\
&\qquad = \angle A'PB' - \angle A'D'P - \angle PC'B' + 180^\circ \\
&\qquad < \angle A'OB' - \angle A'D'O - \angle OC'B' + 180^\circ \\
&\qquad = 180^\circ .
\end{aligned}$$

Hence, B, P, and D are not collinear. Thus, P is not the intersection of AC and BD, and we are done.

Editor's comment: Neither of the submitted solutions addressed the possibility that $C'PB'B$ or $D'PA'D$ might not be convex. For the former, non-convexity means that B and P are in the same half-plane determined by $C'B'$; that is, $\angle PC'B' > 90^\circ$ (because $\angle PC'B = 90^\circ$). The above argument becomes valid with only minor modifications: we must replace $\angle B'PB$ in the sum on the left by $-\angle BPB'$, which equals $-\angle BC'B'$ (angles inscribed in a circle are equal), which in turn equals $-(\angle PC'B' - 90^\circ)$, as in the third line of the final set of equations in the earlier argument. A similar modification works for the expressions involving D and D'.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA.

2948★. [2004 : 231, 233] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Prove or disprove that the unique solution of the system of equations

$$\begin{aligned}
bb' + cc' &= aa' - rr' , \\
a^2 = b^2 + c^2 , &\qquad a'^2 = b'^2 + c'^2 , \\
2r = b + c - a , &\qquad 2r' = b' + c' - a' ,
\end{aligned}$$

among Heron right triangles, where r and r' are their associated inradii, is given by

$$a = 5 ,\ b = 4 ,\ c = 3 ,\ r = 1 ; \quad \text{and} \quad a' = 13 , b' = 12 ,\ c' = 5 ,\ r' = 2 .$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

The system does not have a unique solution, since dilating either or both of the triangles by an integer factor will produce another solution.

This tells us that the problem can be reduced to finding all pairs of primitive Pythagorean triangles that satisfy $bb' + cc' = aa' - rr'$. It is well known that b and c are the lengths of the legs of a primitive Pythagorean triangle if and only if there exist relatively prime positive integers m, n, with different parity, such that $m > n$ and $\{b,\, c\} = \{m^2 - n^2,\, 2mn\}$.

Thus, let $\{b,\, c\} = \{m^2 - n^2,\, 2mn\}$, where m and n are two relatively prime positive integers of different parity and $m > n$. Similary, let $\{b',\, c'\} = \{x^2 - y^2,\, 2xy\}$, where x and y are two relatively prime positive integers of different parity and $x > y$. Then $a = m^2 + n^2$, $r = mn - n^2$, $a' = x^2 + y^2$, $r' = xy - y^2$.

There are two essentially different cases that we need to consider.

Case 1: $(b, c, b', c') = (m^2 - n^2, 2mn, x^2 - y^2, 2xy)$.

Then the equation $bb' + cc' = aa' - rr'$ is equivalent to

$$(m^2-n^2)(x^2-y^2)+4mnxy \;=\; (m^2+n^2)(x^2-y^2)-(mn-n^2)(xy-y^2)\,.$$

When factored, the above equation becomes

$$(nx + ny - 2my)(my + ny - 2nx) \;=\; 0\,.$$

Hence, either $nx + ny = 2my$ or $my + ny = 2nx$. Since these two cases are symmetric, we will assume that the former is true. Thus,

$$n(x + y) \;=\; 2my\,.$$

Since $\gcd(m, n) = \gcd(x+y, 2y) = 1$, we must have $n = 2y$ and $m = x+y$. Therefore, $(m, n, x, y) = (x+y, 2y, x, y)$ (note that $m > n$, $\gcd(m, n) = 1$, and $m + n \equiv 1 \pmod 2$).

Case 2: $(b, c, b', c') = (m^2 - n^2, 2mn, 2xy, x^2 - y^2)$.

Then $bb' + cc' = aa' - rr'$ is equivalent to

$$2(m^2-n^2)xy+2mn(x^2-y^2) \;=\; (m^2+n^2)(x^2-y^2)-(mn-n^2)(xy-y^2)\,,$$

which factors into

$$(3ny + my + nx - mx)(mx - my - nx) \;=\; 0\,.$$

If $3ny+my+nx-mx = 0$, then $4ny = (m-n)(x-y)$, which is impossible since the left side is even and the right side is odd. Hence, $mx-my-nx = 0$, which is equivalent to

$$(m - n)(x - y) \;=\; ny\,.$$

Again, since $\gcd(m - n, n) = \gcd(x - y, y) = 1$, we have $m - n = y$ and $x - y = n$. Thus, $(m, n, x, y) = (x, x - y, x, y)$ (note that x must be even).

All solutions to the original problem can be generated from the aforementioned results, with trivial permutations and dilations. [*Ed.:* For any primitive Pythagorean triangle, calculate the corresponding x and y. Find the second triangle using the results. All such pairs are thus constructed.]

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

2949★. [2004 : 231, 234] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 3$ be an *odd* natural number. Determine the smallest number $\mu = \mu(n)$ such that the entries of any row and of any column of the matrix

$$\begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,\mu} \\ 2 & a_{2,2} & \cdots & a_{2,\mu} \\ \vdots & \vdots & \ddots & \vdots \\ n & a_{n,2} & \cdots & a_{n,\mu} \end{pmatrix}$$

are distinct numbers from the set $\{1, 2, \ldots, n-1, n\}$, and the numbers in each row sum to the same value.

Editor's Comment:

No solutions were received for this problem. However, we were informed by Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON, that they have obtained some partial results. Specifically, they have proved that:

1. $\mu(n) \geq 3$ for all odd $n \geq 3$;

2. $\mu(3) = \mu(7) = \mu(9) = \mu(11) = 3$;

3. $\mu(5) = 5$;

4. $\mu(n) = 3$ for all odd n which are divisible by 3.

Proofs will not be given yet, since Wang and Zhao are continuing their work on the problem.

The proposer, in his submission of the problem, suggested that perhaps $\mu(n) = n$ for all odd n. The partial results are enough to show that this is false. Wang and Zhao believe that $\mu(n) = 3$ for all odd n except $n = 5$.

2950★. [2004 : 231, 234] *Proposed by Walther Janous, Ursulinen-gymnasium, Innsbruck, Austria.*

Let ABC be a triangle whose largest angle does not exceed $2\pi/3$. For λ, $\mu \in \mathbb{R}$, consider inequalities of the form

$$\cos\left(\frac{A}{2}\right)\cdot\cos\left(\frac{B}{2}\right)\cdot\cos\left(\frac{C}{2}\right) \;\geq\; \lambda+\mu\cdot\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)\,.$$

(a) Prove that $\lambda_{\text{max}} \geq \frac{2\sqrt{3}-1}{8}$.

(b) Prove or disprove that

$$\lambda \;=\; \frac{2\sqrt{3}-1}{8} \qquad \text{and} \qquad \mu \;=\; 1+\sqrt{3}$$

yield the best inequality in the sense that λ cannot be increased. Determine also the cases of equality.

Editor's remark: There were no solutions submitted for this problem. As a result, it remains open.