

THE OLYMPIAD CORNER

No. 246

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We begin this number with problems of the Singapore Mathematical Olympiad 2002 (Open Section), written 30 May 2002. My thanks go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining them for our use.

SINGAPORE MATHEMATICAL OLYMPIAD 2002

Open Section

May 30, 2002—Part A

No calculators are allowed. Each question carries a weight of 4 marks. No steps are needed to justify your answers.

- 1.** Let $f(x)$ be a function which satisfies

$$f(29 + x) = f(29 - x),$$

for all values of x . If $f(x)$ has exactly three real roots α , β , and γ , determine the value of $\alpha + \beta + \gamma$.

- 2.** John left town A at x minutes past 6:00 pm and reached town B at y minutes past 6:00 pm the same day. He noticed that at both the beginning and the end of the trip, the minute hand made the same angle of 110 degrees with the hour hand on his watch. How many minutes did it take John to go from town A to town B ?

- 3.** Let $x_1 = \frac{1}{2002}$. For $n \geq 1$, define $nx_{n+1} = (n + 1)x_n + 1$. Find x_{2002} .

- 4.** For integers $n \geq 1$, let $a_n = n^2 + 500$ and $d_n = \gcd(a_n, a_{n+1})$. Determine the largest value of d_n .

- 5.** It is given that the polynomial $p(x) = x^3 + ax^2 + bx + c$ has three distinct positive integer roots and $p(2002) = 2001$. Let $q(x) = x^2 - 2x + 2002$. It is also given that the polynomial $p(q(x))$ has no real roots. Determine the value of a .

- 6.** Find the largest positive integer N such that $N!$ ends with exactly twenty-five “zero” digits.

- 7.** A circle passes through the vertex C of a rectangle $ABCD$ and touches its sides AB and AD at points M and N , respectively. Suppose the distance from C to MN is 2 cm. Find the area of $ABCD$ in cm^2 .

8. Let $m = 144^{\sin^2 x} + 144^{\cos^2 x}$. How many such m 's are integers?

9. Evaluate $\sum_{k=1}^{2002} \frac{k \cdot k!}{2^k} - \sum_{k=1}^{2002} \frac{k!}{2^k} - \frac{2003!}{2^{2002}}$.

10. How many ways are there to arrange 5 identical red, 5 identical blue, and 5 identical green marbles in a straight line such that every marble is adjacent to at least one marble of the same colour as itself?

Part B

Each question carries a weight of 15 marks. Show the steps in your calculations.

1. In the plane, Γ is a circle with centre O and radius r , P and Q are distinct points on Γ , A is a point outside Γ , M and N are the mid-points of PQ and AO , respectively. Suppose $OA = 2a$ and $\angle PQA$ is a right angle. Find the length of MN in terms of r and a . Express your answer in its simplest form, and justify your answer.

2. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be real numbers between 1001 and 2002 inclusive. Suppose $\sum_{i=1}^n a_i^2 = \sum_{i=1}^n b_i^2$. Prove that

$$\sum_{i=1}^n \frac{a_i^3}{b_i} \geq \frac{17}{10} \sum_{i=1}^n a_i^2.$$

Determine when equality holds. _____

3. Let n be a positive integer. Determine the smallest possible sum

$$a_1 b_1 + a_2 b_2 + \dots + a_{2n+2} b_{2n+2},$$

where $a_1, a_2, \dots, a_{2n+2}$ and $b_1, b_2, \dots, b_{2n+2}$ are rearrangements of the binomial coefficients

$$\binom{2n+1}{0}, \binom{2n+1}{1}, \dots, \binom{2n+1}{2n+1}.$$

Justify your answer.

4. Find all real-valued functions $f : \mathbb{Q} \rightarrow \mathbb{R}$ defined on the set of all rational numbers \mathbb{Q} satisfying the conditions

$$f(x+y) = f(x) + f(y) + 2xy,$$

for all x, y in \mathbb{Q} and $f(1) = 2002$. Justify your answers.

The final problems we offer for the April 2005 corner is the XVIII Italian Mathematical Olympiad, Cesenatico, May 3rd, 2002. Thanks again go to Bill Sands for obtaining them.

XVIII ITALIAN MATHEMATICAL OLYMPIAD
Cesenatico, Italy
May 3, 2002

- 1.** Find all 3–digit positive integers that are 34 times the sum of their digits.
- 2.** The plan of a house has the shape of a capital L , obtained by suitably placing side–by–side four squares whose sides are 10 metres long. The external walls of the house are 10 metres high. The roof of the house has six faces, starting at the top of the six external walls, and each face forms an angle of 30° with respect to a horizontal plane.
 Determine the volume of the house (that is, of the solid delimited by the six external walls, the six faces of the roof, and the base of the house).
- 3.** Let A and B be two points of the plane, and let M be the mid–point of AB . Let r be a line, and let R and S be the projections of A and B onto r . Assuming that A , M , and R are not collinear, prove that the circumcircle of triangle AMR has the same radius as the circumcircle of BSM .
- 4.** Find all values of n for which all solutions of the equation $x^3 - 3x + n = 0$ are integers.

- 5.** Prove that, if $m = 5^n + 3^n + 1$ is prime, then 12 divides n .
- 6.** We are given a chessboard with 100 rows and 100 columns. Two squares of the board are said to be *adjacent* if they have a common side. Initially, all squares are white.
 - (a) Is it possible to colour an odd number of squares in such a way that each coloured square has an odd number of adjacent coloured squares?
 - (b) Is it possible to colour some squares in such a way that an odd number of them have exactly 4 adjacent coloured squares and all the remaining coloured squares have exactly 2 adjacent coloured squares?
 - (c) Is it possible to colour some squares in such a way that an odd number of them have exactly 2 adjacent coloured squares and all the remaining coloured squares have exactly 4 adjacent coloured squares?

Next we give a comment on a problem from the XV Gara Nazionale di Matematica 1999 [2002 : 481; 2005 : 37].

2. A natural number is said to be *balanced* if the number of its decimal digits equals the number of its distinct prime factors (for instance 15 is balanced, whereas 49 is not balanced). Prove that there are only finitely many balanced numbers.

Comment by Stan Wagon, Macalester College, St. Paul, MN, USA.

This problem makes me wonder what is the largest balanced number. I believe it is 9592993410, a product of the primes with indices 1, 2, 3, 4, 5, 6, 7, 8, 9, and 14. This number can be found by an easy search once it is known that the answer has at most 10 digits, which can be gleaned from the published solution.

Now, how many balanced numbers are there?

We turn to readers' solutions to problems of the 2000 Hungarian Mathematical Olympiad, given in April, 2003 [2003 : 150].

1. Consider the number of positive even divisors for each of the first n positive integers, and form the sum of these numbers. Form a similar sum of the numbers of positive odd divisors of the first n positive integers. Prove that the two sums differ by at most n .

Solution by Mohammed Aassila, Strasbourg, France.

We will prove that $0 \leq \sum_{k=1}^n o(k) - \sum_{k=1}^n e(k) \leq n$, where $o(k)$ and $e(k)$ denote the number of positive divisors of k which are odd and the number which are even, respectively.

We know that the number of integers divisible by d among $1, 2, \dots, n$ is $\lfloor \frac{n}{d} \rfloor$. Hence,

$$\sum_{k=1}^n o(k) = \sum_{d \text{ odd}} \lfloor \frac{n}{d} \rfloor = \sum_{i=1}^{\infty} \lfloor \frac{n}{2i-1} \rfloor$$

$$\text{and } \sum_{k=1}^n e(k) = \sum_{d \text{ even}} \lfloor \frac{n}{d} \rfloor = \sum_{i=1}^{\infty} \lfloor \frac{n}{2i} \rfloor.$$

Since $\lfloor \frac{n}{d} \rfloor \geq \lfloor \frac{n}{d+1} \rfloor$, we have

$$\sum_{k=1}^n o(k) - \sum_{k=1}^n e(k) = \sum_{i=1}^{\infty} \left(\lfloor \frac{n}{2i-1} \rfloor - \lfloor \frac{n}{2i} \rfloor \right) \geq 0$$

$$\text{and } \sum_{k=1}^n o(k) - \sum_{k=1}^n e(k) = \lfloor \frac{n}{1} \rfloor - \sum_{i=1}^{\infty} \left(\lfloor \frac{n}{2i} \rfloor - \lfloor \frac{n}{2i+1} \rfloor \right) \leq n.$$

2. Construct the point P inside a given triangle such that the feet of the perpendiculars from P to the sides of the triangle determine a triangle whose centroid is P .

Solution by Michel Bataille, Rouen, France.

Let the given triangle be ABC , and let D , E , and F denote the feet of the perpendiculars from P to the sides BC , CA , and AB , respectively.

First, suppose that P is the centroid of $\triangle DEF$. Then the areas $[PEF]$, $[PFD]$, and $[PDE]$ are equal. Hence,

$$PE \cdot PF \cdot \sin A = PF \cdot PD \cdot \sin B = PD \cdot PE \cdot \sin C.$$

(Note that $\angle EPF = 180^\circ - A$, since A, P, E, F all lie on the circle with diameter AP , and similarly, $\angle DPF = 180^\circ - B$ and $\angle EPD = 180^\circ - C$.) It follows that

$$\frac{PE}{PD} = \frac{\sin B}{\sin A} = \frac{b}{a} \quad \text{and} \quad \frac{PF}{PE} = \frac{\sin C}{\sin B} = \frac{c}{b},$$

where, as usual, $a = BC$, $b = CA$, $c = AB$.

Thus, $\frac{PD}{a} = \frac{PE}{b} = \frac{PF}{c}$, and P is the point inside $\triangle ABC$ such that the distances $d(P, BC)$, $d(P, CA)$, $d(P, AB)$ are proportional to a , b , c , respectively. This point is the well-known Lemoine point of $\triangle ABC$.

Conversely, if P is the Lemoine point of $\triangle ABC$, then we have $\frac{PD}{a} = \frac{PE}{b} = \frac{PF}{c} = \lambda$, which implies that

$$[PEF] = PE \cdot PF \cdot \sin A = \lambda^2 bc \frac{a}{2R} = \frac{\lambda^2}{2R} abc$$

(where R is the circumradius of $\triangle ABC$). Thus, $[PEF] = [PFD] = [PDE]$, and P is the centroid of $\triangle DEF$.

To construct P , note that any point M on the median AA' , where A' is the mid-point of BC , satisfies

$$\frac{d(M, AB)}{d(M, AC)} = \frac{d(A', AB)}{d(A', AC)} = \frac{b}{c}.$$

(The latter follows because $[A'AB] = [A'AC]$.)

Thus, if S_A is the reflection of AA' in the internal bisector of $\angle BAC$, then, for any point M' of S_A , we have $\frac{d(M', AB)}{c} = \frac{d(M', AC)}{b}$. The line S_A is the symmedian through A . Constructing similarly the symmedian S_B through B , we obtain P at the intersection of S_A and S_B .

Now we look at solutions from our readers to problems of the 2000 Iranian Mathematical Olympiad given [2003 : 150–151].

2. Triangles $A_3A_1O_2$ and $A_1A_2O_3$ are constructed outside triangle $A_1A_2A_3$, with $O_2A_3 = O_2A_1$ and $O_3A_1 = O_3A_2$. A point O_1 is outside $A_1A_2A_3$ such that $\angle O_1A_3A_2 = \frac{1}{2}\angle A_1O_3A_2$ and $\angle O_1A_2A_3 = \frac{1}{2}\angle A_1O_2A_3$, and T is the foot of the perpendicular from O_1 to A_2A_3 . Prove that:

(a) A_1O_1 is perpendicular to O_2O_3 ;

(b) $\frac{A_1O_1}{O_2O_3} = 2\frac{O_1T}{A_2A_3}$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Let $X \mapsto X'$ be the linear transformation that rotates a vector X counterclockwise by 90° . Let M and N be the mid-points of A_1A_3 and A_1A_2 , respectively. Let $\theta = \angle O_1A_2A_3$ and $\varphi = \angle O_1A_3A_2$. Then $\angle A_3O_2M = \angle A_1O_2M = \theta$ and $\angle A_2O_3N = \angle A_1O_3N = \varphi$. Let $a = \cot \theta$, $b = \cot \varphi$, and $c = a + b$. Let $\mathbf{P} = \overrightarrow{A_1N} = \overrightarrow{NA_2}$, $\mathbf{Q} = \overrightarrow{A_1M} = \overrightarrow{MA_3}$, $\mathbf{R} = \overrightarrow{TO_1}$, $\mathbf{H} = \overrightarrow{A_1T}$, and $\mathbf{V} = \overrightarrow{NM}$.

Now $\overrightarrow{MO_2} = a\mathbf{Q}'$, $\overrightarrow{O_3N} = b\mathbf{P}'$, $\overrightarrow{A_2T} = a\mathbf{R}'$, and $\overrightarrow{TA_3} = b\mathbf{R}'$. Note that $(a+b)\mathbf{R}' = 2\mathbf{V}$; that is, $\mathbf{V} = \frac{c}{2}\mathbf{R}'$. Also, since $A_2T : TA_3 = a : b$, we have

$$\mathbf{H} = \frac{a}{a+b}(2\mathbf{Q}) + \frac{b}{a+b}(2\mathbf{P}) = \frac{2}{c}(a\mathbf{Q} + b\mathbf{P}).$$

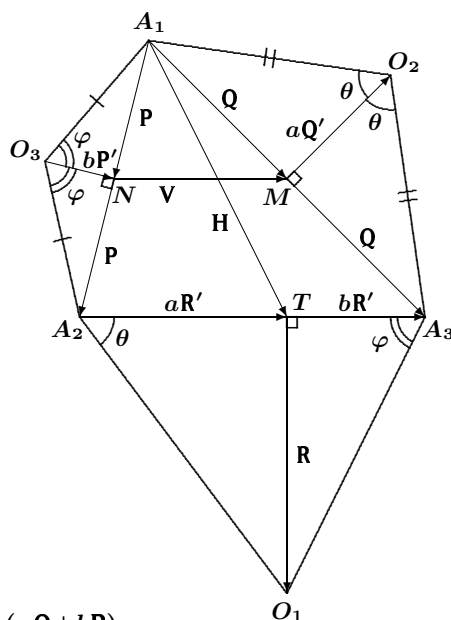
Thus, $a\mathbf{Q} + b\mathbf{P} = \frac{c}{2}\mathbf{H}$.

Now, $\overrightarrow{A_1O_1} = \mathbf{H} + \mathbf{R}$ and

$$\overrightarrow{O_3O_2} = b\mathbf{P}' + \mathbf{V} + a\mathbf{Q}' = (a\mathbf{Q} + b\mathbf{P})' + \mathbf{V} = \frac{c}{2}\mathbf{H}' + \frac{c}{2}\mathbf{R}' = \frac{c}{2}(\overrightarrow{A_1O_1})'.$$

This proves assertion (a).

Also, $A_1O_1 : O_2O_3 = 2 : c$ and $2O_1T : A_2A_3 = 2|\mathbf{R}| : c|\mathbf{R}'| = 2 : c$, which proves (b).



5. Suppose a , b , and c are real numbers such that for any positive real numbers x_1, x_2, \dots, x_n ,

$$\left(\frac{1}{n} \sum_{i=1}^n x_i\right)^a \cdot \left(\frac{1}{n} \sum_{i=1}^n x_i^2\right)^b \cdot \left(\frac{1}{n} \sum_{i=1}^n x_i^3\right)^c \geq 1.$$

Prove that the vector (a, b, c) can be represented as a non-negative linear combination of the vectors $(-2, 1, 0)$ and $(1, -2, 1)$.

Solved by Mohammed Aassila, Strasbourg, France; Robert Bilinski, Outremont, QC; and Pierre Bornshtein, Maisons-Laffitte, France. We give Aassila's write-up.

First set $n = 1$. Then $x_1^{a+2b+3c} = (x_1)^a(x_1^2)^b(x_1^3)^c \geq 1$ for all $x_1 > 0$. Since this is true for both $x_1 > 1$ and $x_1 < 1$, we must have $a + 2b + 3c = 0$. Now,

$$(b + 2c)(-2, 1, 0) + c(1, -2, 1) = (-2b - 3c, b, c) = (a, b, c).$$

It remains to show that $b + 2c \geq 0$ and $c \geq 0$.

To prove that $b + 2c \geq 0$, set $n = 2$, $x_1 = 1$, and $x_2 = t > 0$. The given inequality becomes

$$\left(\frac{1+t}{2}\right)^a \left(\frac{1+t^2}{2}\right)^b \left(\frac{1+t^3}{2}\right)^c \geq 1.$$

Letting $t \rightarrow 0$, we obtain $\frac{1}{2^{a+b+c}} \geq 1$. Hence, $a + b + c \leq 0$ and

$$b + 2c = (a + 2b + 3c) - (a + b + c) \geq 0.$$

To prove that $c \geq 0$, set $n = k + 1$, $x_1 = x_2 = \dots = x_k = 1 - t$, and $x_{k+1} = 1 + kt$, where k is a positive integer and $t \in (0, 1)$. Then

$$\begin{aligned} \sum_{i=1}^n x_i &= k + 1, \\ \sum_{i=1}^n x_i^2 &= (k + 1)(1 + kt^2), \\ \sum_{i=1}^n x_i^3 &= (k + 1)(1 + 3kt^2 + (k^2 - k)t^3). \end{aligned}$$

The given inequality becomes

$$1^a(1 + kt^2)^b(1 + 3kt^2 + k(k - 1)t^3)^c \geq 1.$$

Now, take $t = 1/\sqrt{k}$. Then $2^b \left(4 + \frac{k-1}{\sqrt{k}}\right)^c \geq 1$. Since this inequality holds for all positive integers k , and $\lim_{k \rightarrow +\infty} \left(4 + \frac{k-1}{\sqrt{k}}\right) = +\infty$, then c must be non-negative, and the proof is complete.

6. Prove that for every positive integer n , there exists a polynomial $p(x)$ with integer coefficients such that $p(1), p(2), \dots, p(n)$ are distinct powers of 2.

Solution by Pierre Bornsztejn, Maisons-Laffitte, France.

We will prove a stronger statement: For every positive integer n , there exists a polynomial $p(x)$ with integer coefficients and degree at most n such that $p(0), p(1), \dots, p(n)$ are distinct powers of 2.

Define an $(n+1) \times (n+1)$ matrix M as follows:

$$M = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & n^2 & \dots & n^n \end{pmatrix}.$$

For convenience, we will index the rows and columns of our matrices and vectors starting with 0 rather than 1. Then the entries of M are $m_{ij} = i^j$, for $i = 0, 1, 2, \dots, n$ and $j = 0, 1, 2, \dots, n$ (with $0^0 = 1$).

Let d denote the determinant of M . Since M is a Van der Monde matrix, we have $d = \prod_{0 \leq i < j \leq n} (i - j)$, by the well-known formula for a Van der Monde determinant. It follows that d is a non-zero integer and M is invertible. The entries of M^{-1} are of the form t_{ij}/d , where t_{ij} is an integer.

Since

$$M \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

we must have

$$M^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore,

$$\sum_{j=0}^n \frac{t_{ij}}{d} = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

Note in particular that the above sum is an integer for all i .

Let $d = 2^a b$, where $a, b \geq 0$ are integers and b is odd. Let ω be the order of 2 in the ring of integers modulo b . Then $2^\omega \equiv 1 \pmod{b}$ and $\omega \geq 1$. For every integer $k \geq 0$, we have $2^{k\omega} \equiv 1 \pmod{b}$, and hence there exists an integer c_k such that $2^{k\omega} = 1 + c_k b$.

Let a_0, a_1, \dots, a_n be such that

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = M^{-1} \begin{pmatrix} 2^a \\ 2^{a+\omega} \\ 2^{a+2\omega} \\ \vdots \\ 2^{a+n\omega} \end{pmatrix}.$$

Then, for all i ,

$$a_i = \sum_{j=0}^n \frac{t_{ij}}{d} 2^{a+j\omega} = \sum_{j=0}^n \frac{t_{ij}}{d} 2^a (1 + c_j b) = 2^a \sum_{j=0}^n \frac{t_{ij}}{d} + \sum_{j=0}^n t_{ij} c_j.$$

Since $\sum_{j=0}^n \frac{t_{ij}}{d}$ is an integer, we see that a_i is an integer for each i .

Therefore, the polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ has integer coefficients. Its degree is at most n , clearly. Moreover, since

$$M \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 2^a \\ 2^{a+\omega} \\ 2^{a+2\omega} \\ \vdots \\ 2^{a+n\omega} \end{pmatrix},$$

we have $p(i) = 2^{a+i\omega}$ for each $i = 0, 1, \dots, n$, and we are done.

Now we turn to the May 2003 number of the *Corner* and readers' solutions to problems proposed and shortlisted for the 2000 International Olympiad in Korea given [2003 : 215–216]. George Evagelopoulos, Athens, Greece has supplied the following solutions, some or all of which may be official solutions. My thanks go to him for providing them.

1. (Brazil) Determine all triples of positive integers (a, m, n) such that $a^m + 1$ divides $(a + 1)^n$.

Solution supplied by George Evagelopoulos, Athens, Greece.

It is clear that a triple of positive integers (a, m, n) is a solution if $a = 1$ or $m = 1$. We now look for solutions with $a > 1$ and $m > 1$. We will make use of the following fact (a consequence of the unique prime factorization of positive integers): for any positive integers u and v ,

$$u \mid v^\ell \implies u \mid (\gcd(u, v))^\ell. \quad (1)$$

Let us first prove that, if $a > 1$ and $a^m + 1$ divides $(a + 1)^n$, then m must be odd. Indeed, if m is even, then $a + 1$ divides $a^m - 1$, and hence, $\gcd(a^m + 1, a + 1)$ must be 1 or 2. It follows from (1) that $a^m + 1$ divides 2^n and, hence, $a^m + 1$ is a power of 2, say $a^m + 1 = 2^s$. We have $s \geq 2$, because $a > 1$. Then $a^m = 2^s - 1 \equiv -1 \pmod{4}$, which is impossible, since a^m is a perfect square.

Now suppose that $a > 1$ and m is an odd integer greater than 1. Then $n > 1$. Let p be a prime that divides m , and let $m = pr$ and $b = a^r$. Since r is odd, we see that $a + 1$ divides $b + 1$. Thus, $b^p + 1 = a^m + 1$ divides $(b + 1)^n$. Therefore, the number $B = (b^p + 1)/(b + 1)$ divides $(b + 1)^{n-1}$. In view of (1), we see that B divides $(\gcd(B, b + 1))^{n-1}$. Since $B > 1$, it follows that $\gcd(B, b + 1) > 1$.

Since p is odd, the Binomial Theorem gives

$$B = \frac{b^p + 1}{b + 1} = \frac{(b + 1 - 1)^p + 1}{b + 1} \equiv p \pmod{(b + 1)} .$$

Thus, $\gcd(B, b + 1)$ divides the prime p ; whence $\gcd(B, b + 1) = p$. Then B divides p^{n-1} , implying that B is a power of p . In particular, p divides $b^p + 1$, which in turn divides $(b + 1)^n$. Therefore, p divides $b + 1$, say $b = kp - 1$. Using the Binomial Theorem again, we have

$$\begin{aligned} b^p + 1 &= (kp - 1)^p + 1 = \left((kp)^p - \dots - \binom{p}{2} (kp)^2 + kp^2 - 1 \right) + 1 \\ &\equiv kp^2 \pmod{(kp)^2} . \end{aligned}$$

Then

$$B = \frac{b^p + 1}{b + 1} = \frac{b^p + 1}{kp} \equiv p \pmod{(kp)^2} ,$$

showing that B is divisible by p but not by p^2 . Since B is a power of p , it follows that $B = p$.

Now, if $p \geq 5$, we have

$$\begin{aligned} \frac{b^p + 1}{b + 1} &= b^{p-1} - b^{p-2} + \dots - b + 1 > b^{p-1} - b^{p-2} \\ &= (b - 1)b^{p-2} \geq 2^{p-2} > p . \end{aligned}$$

Thus, we must have $p = 3$, and $B = p$ translates into $b^2 - b + 1 = 3$. This gives $a^r = b = 2$, leading to $a = 2$ and $m = p = 3$. It is immediate that $(2, 3, n)$ is a solution for each $n > 2$.

The set of solutions is, therefore,

$$\{(a, m, n) \mid a = 1 \text{ or } m = 1 \text{ or } (a = 2, m = 3 \text{ and } n \geq 2)\} .$$

2. (Bulgaria) Prove that there exist infinitely many positive integers n such that $p = nr$, where p and r are respectively the semiperimeter and the inradius of a triangle with integer side lengths.

Solved by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and by George Evagelopoulos, Athens, Greece. We give the write-up by Diaz-Barrero, modified by the editor.

Consider an arbitrary triangle with sides a , b , and c , semiperimeter $p = \frac{1}{2}(a + b + c)$, and inradius r . Let $x = b + c - a$, $y = a - b + c$, and $z = a + b - c$. Note that $x = 2(p - a)$, $y = 2(p - b)$, and $z = 2(p - c)$. The area of the triangle is given by $\sqrt{p(p - a)(p - b)(p - c)}$ (Heron's Formula) and also by pr . Therefore, $pr^2 = (p - a)(p - b)(p - c)$; that is, $8pr^2 = xyz$. Hence, $(x + y + z)^3 = (p/r)^2xyz$. Thus, for any integer n , the relation $p = nr$ is equivalent to

$$(x + y + z)^3 = n^2xyz. \quad (1)$$

Now let $n \in \mathbb{N}$ such that there exist positive integers x , y , and z that satisfy equation (1). We can assume that x , y , and z are even, because the equation is homogeneous in x , y , and z . Letting $a = \frac{1}{2}(y + z)$, $b = \frac{1}{2}(x + z)$, and $c = \frac{1}{2}(x + y)$, we see that a , b , and c are positive integers which are the sides of a triangle such that $p = nr$. Thus, the statement in the problem will be proved if we show that equation (1) has positive integer solutions (x, y, z) for infinitely many positive integers n .

We restrict our search for solutions by assuming that $z = k^2(x + y)$ and $n = 3(k^2 + 1)$ for some $k \in \mathbb{N}$. Then equation (1) becomes

$$(x + y)^2(k^2 + 1) = 9k^2xy. \quad (2)$$

Setting $u = x/y$, we obtain $(u + 1)^2(k^2 + 1) = 9k^2u$, or

$$(k^2 + 1)u^2 - (7k^2 - 2)u + (k^2 + 1) = 0.$$

The discriminant of the quadratic on the left side above is

$$\Delta = (7k^2 - 2)^2 - 4(k^2 + 1) = 9k^2(5k^2 - 4).$$

Now u will be rational if and only if Δ is a square, and this will be true if and only if there is some $\ell \in \mathbb{N}$ such that

$$5k^2 - 4 = \ell^2. \quad (3)$$

We claim that equation (3) has solutions (k_j, ℓ_j) , for $j \in \mathbb{N}$, defined recursively by $(k_1, \ell_1) = (1, 1)$ and, for $j \geq 1$,

$$(k_j, \ell_j) = \left(\frac{3k_{j-1} + \ell_{j-1}}{2}, \frac{5k_{j-1} + 3\ell_{j-1}}{2} \right).$$

In fact, $(1, 1)$ is a solution, and it is easy to check that (k_j, ℓ_j) is a solution whenever (k_{j-1}, ℓ_{j-1}) is a solution. Thus, the claim follows by induction.

An easy induction shows that $k_{j-1} < k_j$ for all j . Thus, the sequence $\{k_j\}_{j=1}^{\infty}$ is strictly increasing. For each integer k in this sequence, there is a rational number $u = x/y$ such that equation (2) is satisfied. Then, for $n = 3(k^2 + 1)$, we have a solution (x, y, z) for equation (2). We have obtained solutions for infinitely many values of n .

3. (Colombia) Let $n \geq 4$ be a fixed integer. A set $S = \{P_1, \dots, P_n\}$ of n points is given in the plane such that no three are collinear and no four concyclic. For $1 \leq t \leq n$, let a_t be the number of circles $P_i P_j P_k$ that contain P_t in their interior, and let $m(S) = a_1 + a_2 + \dots + a_n$. Prove that there exists a positive integer $f(n)$, depending only on n , such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

Solution supplied by George Evagelopoulos, Athens, Greece.

We prove that $m(S) \leq 2\binom{n}{4}$, with equality if and only if the points of S are the vertices of a convex polygon. The proof is by induction on n .

First we prove the assertion for the case $n = 4$. We consider two cases.

Case (i). The points are not the vertices of a convex quadrilateral.

In this case, we may assume that P_4 lies inside the triangle $P_1 P_2 P_3$. Then $a_1 = a_2 = a_3 = 0$ and $a_4 = 1$; whence, $m(S) = 1 < 2\binom{4}{4}$.

Case (ii). The points P_1, P_2, P_3 , and P_4 are consecutive vertices of a convex quadrilateral.

This quadrilateral is not cyclic (according to one of the restrictions on S). If $\angle P_1 + \angle P_3 > 180^\circ$, then $\angle P_2 + \angle P_4 < 180^\circ$ and it is easily checked that $a_1 = a_3 = 1$ and $a_2 = a_4 = 0$, giving $m(S) = 2 = 2\binom{4}{4}$. Otherwise, we must have $\angle P_1 + \angle P_3 < 180^\circ$ and $\angle P_2 + \angle P_4 > 180^\circ$, which again gives us $m(S) = 2 = 2\binom{4}{4}$.

Now assume that the claim is true for $n = k \geq 4$, and consider a set $S = \{P_1, P_2, \dots, P_{k+1}\}$ of $k + 1$ points in general position in the plane. For each $i = 1, \dots, k + 1$, let $S_i = S \setminus \{P_i\}$. We compute in two different ways the number N of possible choices of five distinct points P_a, P_b, P_c, P_d, P_e from S such that the circle $P_b P_c P_d$ contains P_a .

There are $m(S)$ choices for the pair $(P_a, P_b P_c P_d)$. Once we have made this choice, there are $k - 3$ choices for P_e . Hence, $N = (k - 3)m(S)$. On the other hand, choosing P_e first, we have $m(S_e)$ choices for the pair $(P_a, P_b P_c P_d)$. Hence, $N = m(S_1) + m(S_2) + \dots + m(S_{k+1})$. Thus,

$$\begin{aligned} m(S) &= \frac{m(S_1) + m(S_2) + \dots + m(S_{k+1})}{k - 3} \\ &\leq \frac{(k + 1)2\binom{k}{4}}{k - 3} = 2\binom{k + 1}{4}. \end{aligned}$$

Here, we have used the induction hypothesis for each of the k -element sets S_1, S_2, \dots, S_{k+1} . Equality holds if and only if the k points of each set S_i form a convex k -gon. It is easily seen that this is true if and only if the points of S form a convex $(k + 1)$ -gon. This completes the proof.

4. (Czech Republic) Let n and k be positive integers such that $\frac{n}{2} < k \leq \frac{2n}{3}$. Determine the least positive integer m for which it is possible to place each of m pawns on a square of an $n \times n$ chessboard so that no column or row contains a block of k adjacent unoccupied squares.

Solution supplied by George Evagelopoulos, Athens, Greece.

Let us say that a placement of pawns on the board is *good* if there is no $k \times 1$ or $1 \times k$ block of unoccupied squares. Label the rows and columns 0 through $n - 1$.

A standard good placement is obtained by putting pawns on all squares (i, j) such that $i + j + 1$ is divisible by k . This pawn pattern consists of diagonal lines of pawns across the board. Note that $i + j + 1 \leq 2n - 1$, and hence, since $n < 2k$, we have $i + j + 1 < 4k - 1$. Therefore, when $i + j + 1$ is divisible by k , we must have $i + j + 1 \in \{k, 2k, 3k\}$. Thus, there are at most three lines of pawns. One line (where $i + j + 1 = k$) contains k pawns, a second (where $i + j + 1 = 2k$) contains $2n - 2k$ pawns, and a third (where $i + j + 1 = 3k$) contains $2n - 3k$ pawns (this line vanishes in the extreme case where $3k = 2n$). In all cases, the total number of squares occupied by pawns is $4(n - k)$.

We now show that $4(n - k)$ is actually the least possible number of pawns in a good placement. Suppose we have a good placement of m pawns. Partition the board into nine rectangular regions

$$\begin{array}{ccc} A & B & C \\ D & E & F \\ G & H & I \end{array}$$

so that the corner regions A , C , G , and I are $(n - k) \times (n - k)$ squares, B and H are $(n - k) \times (2k - n)$ rectangles, and D and F are $(2k - n) \times (n - k)$ rectangles. This is possible, since $2k - n > 0$. Assume that there are exactly b rows free of pawns in the rectangle B , h rows free in H , d columns free in D , and f columns free in F .

Take any one of the b rows that are pawn-free in B , and extend it to the left and to the right across the whole board. The portion of the extended row within rectangle A must contain at least one occupied square; otherwise the placement would not be good. The same can be said about its portion within C . Choose one occupied square in A and one occupied square in C (in that row) and put markers on those two squares.

Do the same for each of the b rows that are pawn-free in B and the h rows that are pawn-free in H . Perform a similar operation on each of the d columns that are pawn-free in D and the f columns that are pawn-free in F . At each step, we have put markers on two squares in $A \cup C \cup G \cup I$. In total, we have distributed $2(b + h + d + f)$ markers. A square could have been marked at most twice. Thus, the number of marked squares is at least $b + h + d + f$, and there is a pawn on each of them. Hence, we have at least that many pawns in $A \cup C \cup G \cup I$.

Moreover, there are at least $n - k - b$ pawns in B (since there is at least one pawn in each of the $n - k$ rows of B that are not pawn-free in B). Likewise, there are at least $n - k - h$ pawns in H , at least $n - k - d$ in D , and at least $n - k - f$ in F . Hence, $m \geq 4(n - k)$. Thus, $4(n - k)$ is the minimum sought.

5. (France) Let p and q be relatively prime positive integers. Determine the number of subsets S of $\{0, 1, 2, \dots\}$ such that $0 \in S$ and, for each element $n \in S$, the integers $n + p$ and $n + q$ belong to S .

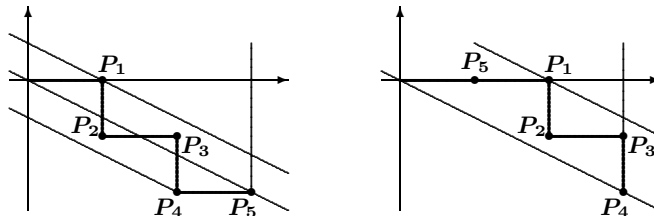
Solution supplied by George Evagelopoulos, Athens, Greece.

Every integer z has a unique representation $z = px + qy$ with integer coefficients x, y such that $0 \leq x < q$. This inequality defines a vertical strip in \mathbb{R}^2 in which every lattice point (x, y) corresponds bijectively to an integer $px + qy$. For each such lattice point, we write the corresponding integer $px + qy$ in the unit square $[x, x + 1) \times [y, y + 1)$. Then every integer appears exactly once, and the non-negative integers are in the squares that correspond to lattice points (x, y) on or above the line $px + qy = 0$.

Let S be a set as described in the problem. We will call any such set *ideal*. All the elements of S have been written in some squares (in the strip in question). Put markers in those squares. We are given that if $n \in S$, then $n + p \in S$ and $n + q \in S$. It follows by induction that if a square Q is marked, then all squares that lie above Q or to the right of Q are also marked. In particular, since $0 \in S$, all squares in the half-strip $0 \leq x < q, y \geq 0$ are marked.

It remains only to consider the squares associated with lattice points in the right triangle Δ defined by $px + qy > 0, y < 0, x < q$. If any such square Q is marked, then the whole portion of Δ upwards and rightwards of Q has to be marked. The border between the marked part of Δ and the unmarked part must be a polygonal path which follows gridlines between lattice points, running south and east from $(0, 0)$ to $(q, -p)$ without ever crossing the line $px + qy = 0$. We call such a path an *ideal path*. All that remains is to count the ideal paths.

Let Γ denote the set of all gridline paths of length $p + q$ from $(0, 0)$ to $(q, -p)$. Then the number of members of Γ equals $\binom{p+q}{p}$. Let E and S denote the unit moves east and south, respectively. Each path $\gamma \in \Gamma$ gives rise to a sequence D_1, D_2, \dots, D_{p+q} , where $D_i \in \{E, S\}$, such that q of the D_i s are E and p of the D_i s are S . For a path $\gamma = D_1, D_2, \dots, D_{p+q}$, and for $i = 1, 2, \dots, p + q$, let P_i be the point, called a *vertex* of γ , reached after tracing D_1, D_2, \dots, D_i from $(0, 0)$, and let ℓ_i be the line that is parallel to $px + qy = 0$ and passes through P_i . Since the correspondence between values of $px + qy$ and lattice points within the strip $0 \leq x < q$ is bijective, we see that the lines $\ell_1, \dots, \ell_{p+q}$ are distinct.



Two paths are said to be equivalent if one is obtained from the other by a circular shift of the coding sequence D_1, D_2, \dots, D_{p+q} . For $\gamma \in \Gamma$, the equivalence class containing γ has $p+q$ elements, because γ admits precisely $p+q$ cyclic shifts and all of them are distinct. If $\gamma = D_1, D_2, \dots, D_{p+q}$, let m be such that ℓ_m is the lowest among the ℓ_i 's. Since the lines ℓ_i are distinct, such an m is unique. Then the path

$$D_m, \dots, D_{p+q}, D_1, \dots, D_{m-1}$$

is above the line $px + qy = 0$. Every other cyclic shift gives rise to a path of which at least one vertex is below the line $px + qy = 0$. Thus, each equivalence class contains exactly one ideal path, implying that the number of ideal paths equals $\frac{1}{p+q} \binom{p+q}{p}$.

6. (France) For a positive integer n , let $d(n)$ be the number of positive divisors of n . Find all positive integers such that $d(n)^3 = 4n$.

Solution supplied by George Evagelopoulos, Athens, Greece.

Let n be a positive integer such that $d(n)^3 = 4n$. For each prime number p , let a_p denote the exponent of p in the prime factorization of n . Since $4n$ is a cube, we must have $a_2 = 1 + 3\beta_2$ and $a_p = 3\beta_p$ for $p \geq 3$, where each β_p is a non-negative integer. Now we use the well-known result $d(n) = \prod_p (1 + a_p)$, where the product is taken over all primes p . We obtain

$$d(n) = (2 + 3\beta_2) \prod_{p \geq 3} (1 + 3\beta_p).$$

This relation shows that 3 does not divide $d(n)$. Since $d(n)^3 = 4n$, we see that 3 does not divide n . Thus, $\beta_3 = 0$, and the equation $d(n)^3 = 4n$ becomes

$$\frac{2 + 3\beta_2}{2^{1+\beta_2}} = \prod_{p \geq 5} \frac{p^{\beta_p}}{1 + 3\beta_p}. \quad (1)$$

For $p \geq 5$, we have $p^{\beta_p} \geq 5^{\beta_p} = (1 + 4)^{\beta_p} \geq 1 + 4\beta_p$. It follows that the right side of (1) is at least 1, and equality holds if and only if $\beta_p = 0$ for all $p \geq 5$. We deduce that $2 + 3\beta_2 \geq 2^{1+\beta_2}$. Hence,

$$2 + 3\beta_2 \geq 2(1 + 1)^{\beta_2} \geq 2 \left(1 + \binom{\beta_2}{1} + \binom{\beta_2}{2} \right) = 2 + \beta_2 + \beta_2^2.$$

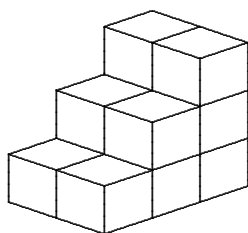
This gives $2\beta_2 \geq \beta_2^2$, implying that $\beta_2 \leq 2$. Thus, $\beta_2 \in \{0, 1, 2\}$.

If $\beta_2 = 0$ or $\beta_2 = 2$, then $\frac{2 + 3\beta_2}{2^{1+\beta_2}} = 1$ and both sides of equation (1) are equal to 1. Hence, 2 is the only prime divisor of n . Thus, we obtain the solutions $n = 2$ and $n = 2^7 = 128$.

If $\beta_2 = 1$, then $\frac{2+3\beta_2}{2^{1+\beta_2}} = \frac{5}{4}$, and there must be a factor of 5 on the right side of equation (1); that is, $\beta_5 > 0$. We cannot have $\beta_5 > 1$, because this would give $\frac{5^{\beta_5}}{1+3\beta_5} > \frac{5}{4}$. Therefore, $\beta_5 = 1$. Then $\frac{5^{\beta_5}}{1+3\beta_5} = \frac{5}{4}$. Substituting in (1) gives $\beta_p = 0$ for all $p \geq 7$. Thus, we obtain the solution $n = 2^4 \cdot 5^3 = 2000$.

In conclusion, the solutions are 2, 128, and 2000.

7. (Italy) The diagram shows a staircase-brick with 3 steps of width 2, made of 12 unit cubes. Determine all positive integers n for which it is possible to build an $n \times n \times n$ cube using such bricks.



Solution supplied by George Evagelopoulos, Athens, Greece.

The volume of a single brick is equal to 12. Thus, a necessary condition is that the side of the cube is a multiple of 6. Two bricks fit together to make a parallelepiped of size $2 \times 3 \times 4$, and such parallelepipeds can be stacked to form a cube of side 12, as well as any multiple of 12. We claim that this last condition is also necessary; that is, a cube of side $n = 6\ell$ can be constructed only if ℓ is even.

Now suppose a cube is constructed using $m = n^3/12 = 18\ell^3$ bricks, and that it is positioned in the octant with $x, y, z \geq 0$ with a vertex at the origin $O = (0, 0, 0)$. Colour each unit cube $[i, i+1] \times [j, j+1] \times [k, k+1]$ with one of eight colours, depending on the parity of i, j , and k in the triple (i, j, k) . In each brick, all the eight colours are present, six of them appearing on only one unit cube and each of the remaining two colours appearing on three unit cubes.

Choose one of the eight colours, and let p be the number of bricks in which this colour appears three times. The number of bricks in which this colour appears only once is then $m - p$. Thus, this colour appears in the cube a total of $3p + (m - p) = m + 2p$ times. On the other hand, the eight colours are equally distributed in the cube, since the cube has sides of even length. Thus, each colour appears exactly $12m/8$ times. It follows that $m + 2p = 12m/8$; whence, $m = 4p$. Therefore, m is a multiple of 4, and ℓ must be even.

That completes the *Corner* for this issue. I wish you good problem solving over the summer. Send me your nice solutions and generalizations.