

# SKOLIAD No. 86

Robert Bilinski

Please send your solutions to the problems in this edition by **1 August, 2005**. A copy of **MATHEMATICAL MAYHEM Vol. 4** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (\*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

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Our items come, once again, from the 4<sup>th</sup> annual CNU Regional Mathematics Contest. We are presenting only a selection of the remaining problems. Thanks go to R. Porsky, C.N.U., Newport News, VA.

## 4<sup>e</sup> Concours Annuel CNU Régional de Mathématique du Secondaire Samedi, le 6 Décembre 2003

- 17.** Le nombre de solutions entières de  $2(x + y) = xy + 7$  est  
 (A) 1            (B) 2            (C) 3            (D) 4
- 20.** Le minimum de  $S = x^2 + 2xy + 3y^2 + 2x + 6y + 4$  est  
 (A) 4            (B) 1            (C) 0            (D) -1
- 22.** Si  $3 \sin \theta + 4 \cos \theta = 5$ , alors  $\tan \theta$  vaut  
 (A) 1            (B) -1            (C)  $\frac{3}{4}$             (D)  $\frac{4}{3}$
- 24.** Si  $f(x + y) = f(xy)$  et  $f(7) = 7$ , alors  $f(49) =$   
 (A) 49            (B) 14            (C) 7            (D) 1
- 26.** Soit  $a = 1! + 2! + 3! + \dots + 2003!$ . Le chiffre des unités de  $a$  vaut  
 (A) 9            (B) 5            (C) 3            (D) 0
- 27.** Trouver la somme de tous les entiers positifs non-multiples de 3 inférieurs à 45.  
 (A) 600            (B) 625            (C) 650            (D) 675

**28.** Quel est le plus petit entier  $k$  tel que  $2x(kx - 4) - x^2 + 6 = 0$  n'ait pas de solution réelle ?

- (A)  $-1$       (B)  $2$       (C)  $3$       (D)  $4$

**29.** Le nombre de facteurs entiers distincts de  $30^4$  est

- (A)  $100$       (B)  $125$       (C)  $123$       (D)  $30$

**30.** La somme des racines de  $f(x) = x(2x + 3)(4x + 5) + (6x + 7)(8x + 9)$  est

- (A)  $-\frac{35}{4}$       (B)  $\frac{35}{4}$       (C)  $-70$       (D)  $70$

**32.** Une ligne  $L$  a une pente de  $-2$  et passe par le point  $(r, -3)$ . Une seconde ligne  $K$ , perpendiculaire à  $L$  en  $(a, b)$ , passe par le point  $(6, r)$ . Le valeur de  $a$  est

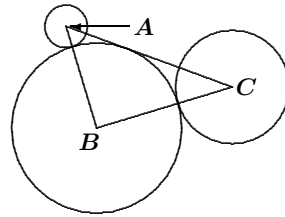
- (A)  $r$       (B)  $\frac{2r}{5}$       (C)  $2r - 3$       (D)  $1$

**38.** Une boîte contient 11 balles, numérotées  $1, 2, \dots, 11$ . Si 6 balles sont pigées simultanément au hasard, quelle est la probabilité que la somme des nombres pigés soit impaire ?

- (A)  $\frac{100}{231}$       (B)  $\frac{115}{231}$       (C)  $\frac{1}{2}$       (D)  $\frac{118}{231}$

**3.** (Questions en équipe) Trouver le maximum de  $f(x) = \left(\frac{1}{2}\right)^{x^2 - 2x}$ .

**10.** (Questions en équipe) Sur le dessin, le cercle centré en  $A$  a un rayon de 1 et le gros cercle centré en  $B$  a un rayon de 4. Le troisième cercle est centré en  $C$ , et le gros cercle touche les deux autres cercles. De plus,  $\angle ABC$  est un angle droit et la ligne  $AC$  touche le gros cercle. Trouver le rayon du cercle centré en  $C$ .



**11.** (Questions en équipe) Un homme voyage en automobile à une vitesse moyenne de 50 miles par heure. Il revient par le même chemin à une vitesse moyenne de 30 miles par heure. Quelle est la vitesse moyenne pour le voyage ?

**12.** (Questions en équipe) Résoudre algébriquement pour  $x$  :

$$(\log_{10} x^2)^2 = \log_{10}(x^4).$$

**4<sup>th</sup> Annual CNU Regional High School  
Mathematics Contest  
Saturday December 6, 2003**

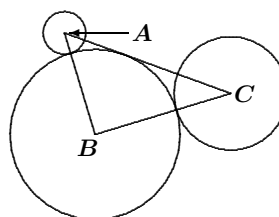
- 17.** The number of positive integer solutions for  $2(x + y) = xy + 7$  is  
(A) 1            (B) 2            (C) 3            (D) 4
- 20.** The minimum of  $S = x^2 + 2xy + 3y^2 + 2x + 6y + 4$  is  
(A) 4            (B) 1            (C) 0            (D) -1
- 22.** If  $3 \sin \theta + 4 \cos \theta = 5$ , then  $\tan \theta$  is  
(A) 1            (B) -1            (C)  $\frac{3}{4}$             (D)  $\frac{4}{3}$
- 24.** If  $f(x + y) = f(xy)$  and  $f(7) = 7$ , then  $f(49) =$   
(A) 49            (B) 14            (C) 7            (D) 1
- 26.** Let  $a = 1! + 2! + 3! + \dots + 2003!$ . Then the units digit of  $a$  is  
(A) 9            (B) 5            (C) 3            (D) 0
- 27.** Find the sum of all the positive integers less than 45 that are not divisible by 3.  
(A) 600            (B) 625            (C) 650            (D) 675
- 28.** What is the smallest integer  $k$  such that  $2x(kx - 4) - x^2 + 6 = 0$  has no real solutions?  
(A) -1            (B) 2            (C) 3            (D) 4
- 29.** The number of distinct positive integer factors of  $30^4$  is  
(A) 100            (B) 125            (C) 123            (D) 30
- 30.** The sum of the zeros of  $f(x) = x(2x + 3)(4x + 5) + (6x + 7)(8x + 9)$  is  
(A)  $-\frac{35}{4}$             (B)  $\frac{35}{4}$             (C) -70            (D) 70
- 32.** A line  $L$  has a slope of -2 and passes through the point  $(r, -3)$ . A second line  $K$  is perpendicular to  $L$  at  $(a, b)$  and passes through the point  $(6, r)$ . The value of  $a$  is  
(A)  $r$             (B)  $\frac{2r}{5}$             (C)  $2r - 3$             (D) 1

**38.** A box contains 11 balls, numbered 1, 2, ..., 11. If 6 balls are drawn simultaneously at random, what is the probability that the sum of the numbers drawn is odd?

- (A)  $\frac{100}{231}$       (B)  $\frac{115}{231}$       (C)  $\frac{1}{2}$       (D)  $\frac{118}{231}$

**3.** (Team Round) Find the maximum value of  $f(x) = \left(\frac{1}{2}\right)^{x^2-2x}$ .

**10.** (Team Round) In the diagram, the circle with centre  $A$  has radius 1, and the big circle with centre  $B$  has radius 4. The third circle has centre  $C$ , and the big circle touches the two other circles. Also,  $\angle ABC$  is a right angle and the line  $AC$  touches the big circle. Find the radius of the circle with centre  $C$ .



**11.** (Team Round) A man makes a trip by automobile at an average speed of 50 miles per hour. He returns over the same route at an average speed of 30 miles per hour. What is his average speed for the entire trip?

**12.** (Team Round) Solve algebraically for  $x$ :

$$(\log_{10} x^2)^2 = \log_{10}(x^4).$$

Next we give the solutions to the 1993–1994 Newfoundland and Labrador Teachers Association, Senior Mathematics League Game 4 [2004 : 449–451].

### 1993–1994 Newfoundland and Labrador Teachers Association Senior Mathematics League Game 4

**1.** (\*) If  $n$  is a positive integer then  $n!$  (read “ $n$  factorial”) is defined to be

$$n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \cdots 3 \cdot 2 \cdot 1.$$

For example  $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$  and  $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ .

Determine the positive integer  $m$  such that the number of seconds in a year is between  $m!$  and  $(m + 1)!$

*Solution by the editor.*

The number of seconds in a year is  $S = 365 \times 24 \times 3600 = 2^7 \cdot 3^3 \cdot 5^3 \cdot 73$ . If we organize the factors of  $S$  to resemble a factorial, we get:

$$\begin{aligned} S &= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8 \cdot 3 \cdot 5^2 \cdot 73 \\ &= 10! \cdot \frac{5 \cdot 73}{2 \cdot 3 \cdot 7} = 10! \cdot \frac{365}{42}. \end{aligned}$$

However,  $1 < \frac{365}{42} < 11$ . Thus,  $10! < S < 11!$ , which implies that  $m = 10$  is the integer we seek.

(Even if we had assumed a leap year and used 366 instead of 365, the answer would have remained the same.)

*Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON. There was one incorrect solution submitted.*

**2.** (\*) A movie showing was attended by 500 people. Adults paid \$10 each and children \$4 each. The total amount taken for tickets was \$4,160. How many children attended?

*Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

An adult pays \$6 more than a child; thus, there are  $4(500) = 2000$  dollars in sales plus an extra \$6*a* in sales, where *a* is the number of adults in the group. We must have  $2000 + 6a = 4160$ . Solving for *a*, we obtain  $a = \frac{4160 - 2000}{6} = 360$ . Hence, there were  $500 - 360 = 140$  children.

*Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON.*

**3.** (\*) Assume that the earth is a sphere with circumference 40,250 km and that a belt is placed around the equator, one metre above the earth's surface at all points. How much greater than the circumference of the earth would the length of the belt be? Would this difference be:

- (a)  $2\pi$  metres,
- (b) 40,250 metres,
- (c)  $40,250\pi$  metres,
- (d) 40,250 kilometres, or
- (e) none of the above?

*Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

Suppose the radius of the earth is *r* km. Then the circumference of the earth is  $C = 40,250 = 2\pi r$  km, while the circumference of the belt in km is clearly  $2\pi(r + 0.001) = C + 0.002\pi$ . Thus, the difference is  $0.002\pi$  km, or  $2\pi$  metres.

*Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON.*

**4.** (\*) Let *a*, *b* and *c* be integers. You are given that  $a \star b$  is defined to be  $ab - 2a - 2b + 6$ . Compute

$$(a \star b) \star c - a \star (b \star c).$$

I. *Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

Equivalently,  $a \star b$  is defined as  $(a - 2)(b - 2) + 2$ . Then

$$\begin{aligned}(a \star b) \star c &= ((a \star b) - 2)(c - 2) + 2 \\ &= ((a - 2)(b - 2) + 2 - 2)(c - 2) + 2 \\ &= (a - 2)(b - 2)(c - 2) + 2.\end{aligned}$$

Similarly,  $a \star (b \star c) = (a - 2)(b - 2)(c - 2) + 2$ . Thus, the difference is zero.

II. *Solution by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON.*

$$\begin{aligned}(a \star b) \star c - a \star (b \star c) &= (ab - 2a - 2b + 6) \star c - a \star (bc - 2b - 2c + 6) \\ &= [(ab - 2a - 2b + 6)c - 2(ab - 2a - 2b + 6) - 2c + 6] \\ &\quad - [a(bc - 2b - 2c + 6) - 2a - 2(bc - 2b - 2c + 6) + 6] \\ &= (abc - 2ac - 2bc + 6c - 2ab + 4a + 4b - 12 - 2c + 6) \\ &\quad - (abc - 2ab - 2ac + 6a - 2a - 2bc + 4b + 4c - 12 + 6) \\ &= 0.\end{aligned}$$

5. (\*) A cube with sides of length 3 cm is painted red and then cut into  $3 \times 3 \times 3 = 27$  cubes with sides of length 1 cm. If  $a$  denotes the number of small cubes (that is,  $1 \text{ cm} \times 1 \text{ cm} \times 1 \text{ cm}$  cubes) that are not painted at all,  $b$  the number painted on one side,  $c$  the number painted on two sides, and  $d$  the number painted on three sides, determine  $a - b - c + d$ .

*Solution by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON.*

The diagram at right shows one face of the painted cube, where the letter on a square shows to which count that small cube contributes. (Remember that each square on the face represents a small cube.)

$d$	$c$	$d$
$c$	$b$	$c$
$d$	$c$	$d$

Clearly  $a = 1$ , since only the small cube in the core of the big cube receives no paint. Also,  $b = 6$  since there are six faces as in the diagram each contributing 1 to  $b$ . There are 4 squares labelled  $c$  in the diagram, but each of the corresponding small cubes is counted twice when we sum over all six faces of the big cube; thus,  $c = \frac{4 \times 6}{2} = 12$ . Finally, there are 4 squares labelled  $d$  in the diagram, and each of the corresponding small cubes is counted three times when we sum; hence,  $d = \frac{4 \times 6}{3} = 8$ . Consequently,  $a - b - c + d = 1 - 6 - 12 + 8 = -9$ .

6. For which value or values of  $k$ , if any, is  $x^2 + k$  a factor of

$$x^4 - 3x^3 + 6x^2 - 3kx + 8?$$

*Solution by Eric Zhang, grade 11, Lisgar Collegiate, Ottawa, ON.*

Suppose that  $x^2 + k$  is a factor. Then the other factor must be of the form  $x^2 + ax + b$ , and we have

$$\begin{aligned}x^4 - 3x^3 + 6x^2 - 3kx + 8 &= (x^2 + k)(x^2 + ax + b) \\ &= x^4 + ax^3 + (b + k)x^2 + akx + kb.\end{aligned}$$

Equating coefficients on the left and right sides, we obtain the system of equations:

$$a = -3, \quad (1)$$

$$b + k = 6, \quad (2)$$

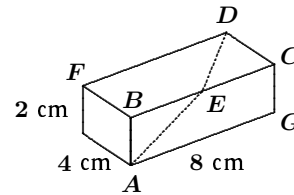
$$kb = 8. \quad (3)$$

From (2), we get  $b = 6 - k$ ; substituting this into (3), we obtain  $k(6 - k) = 8$ ; that is,  $(k - 4)(k - 2) = 0$ . Therefore,  $k = 2$  or  $k = 4$ .

*Also solved by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

**7.** An ant wishes to travel from  $A$  to  $D$  on the surface of a small wooden block with dimensions 2 cm by 4 cm by 8 cm, as shown on the right. The shortest such route involves crossing the edge  $BC$  at a point  $E$ .

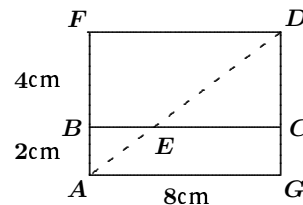
Find the distance  $BE$ .



*Solution by Eric Zhang, grade 11, Lisgar Collegiate, Ottawa, ON.*

By unfolding the block as shown, we see that for  $DEA$  to be the shortest route, it must be a straight line. Observe that the two right triangles  $DAF$  and  $EAB$  are similar. Thus,  $\frac{BE}{FD} = \frac{AB}{AF}$ , which implies that

$$BE = \frac{AB}{AF} \times FD = \frac{2}{6} \times 8 = \frac{8}{3} \text{ cm.}$$



*Also solved by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

**8.** A typical large hamburger has 427 calories, 48% of them from fat. The same hamburger with cheese has 31 grams of fat, 53% of its calories coming from fat. Regular French fries have 220 calories in total and 12 grams of fat. A popular sundae has 360 calories, and 28% of these are fat calories.

A high school student has a meal consisting of this hamburger with double cheese, an order of regular French fries and the popular sundae. What percentage of calories in the meal are fat calories?

You need to know that 1 gram of fat has 9 calories. Give your answer to the nearest whole number percentage.

*Solution by Eric Zhang, grade 11, Lisgar Collegiate, Ottawa, ON, modified by the editor.*

Let the pair  $(t, f)$  represent  $t$  total calories and  $f$  fat calories. Let  $HB$  be the calorie pair for a hamburger. In the same manner, let  $CB$  represent a Cheeseburger,  $FF$  an order of regular French fries,  $S$  a sundae,  $C$  a piece of cheese, and  $DCB$  a double cheeseburger. We have

$$\begin{aligned} HB &= (427, 48\% \times 427) = (427, 204.96), \\ CB &= \left( \frac{31 \times 9}{53\%}, 31 \times 9 \right) \approx (526.42, 279), \\ C &= CB - HB \approx (99.42, 74.04), \\ DCB &= CB + C \approx (625.83, 353.04), \\ FF &= (220, 12 \times 9) = (220, 108), \\ S &= (360, 28\% \times 360) = (360, 100.8). \end{aligned}$$

Hence, if  $M$  represents the calorie pair for the meal in question, then

$$M = DCB + FF + S \approx (1205.83, 561.84).$$

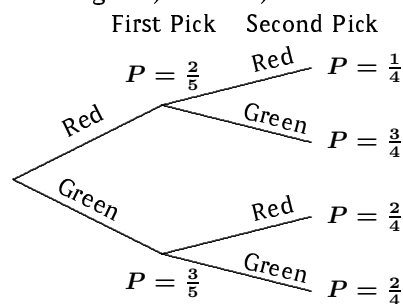
Therefore, the fraction of calories which are fat calories in the meal is  $561.84/1205.83$ , which is approximately 47%.

*One incorrect solution was received.*

**9.** A bag contains 2 red cabbages and 3 green cabbages. Tracy, who is blind-folded, randomly selects one of the cabbages and places it in an empty pan. Then she randomly selects a second cabbage from those remaining in the bag and also places that in the pan. What is the percentage likelihood that, of the two cabbages that are now in the pan, one is red and the other is green?

*Composite of solutions by Alex Wice, grade 11 student, Leaside High School, Toronto, ON; and Eric Zhang, grade 11, Lisgar Collegiate, Ottawa, ON.*

We can achieve the desired outcome by selecting either red followed by green or green followed by red (see the diagram). The first possibility yields the probability of success as  $\frac{2}{5} \cdot \frac{3}{4} = \frac{3}{10}$ ; the second possibility gives the probability of success as  $\frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$ . Summing gives the total probability as  $\frac{6}{10}$ , which is a 60% likelihood.

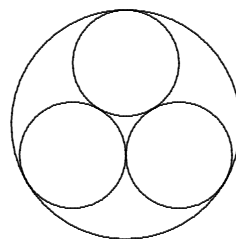


[*Ed.:* An alternate method that avoids decision trees is also available. The total number of possible ways to pick two cabbages from the bag is  $\binom{5}{2} = 10$ . Tracy needs to pick one of the two red cabbages, which can be done in  $\binom{2}{1} = 2$  ways, and one of the three green cabbages, which can be done in  $\binom{3}{1} = 3$  ways. Thus, the probability of success is  $\frac{2+3}{10}$ , or 60%.]



**10.** Three small circles each of radius 1 cm and one larger circle are located as indicated on the right. Determine the area of the larger circle.

Your answer should be expressed in the form  $\left(\frac{a + b\sqrt{3}}{c}\right)\pi$ , where  $a$ ,  $b$ , and  $c$  are integers.



*Solution by Eric Zhang, grade 11, Lisgar Collegiate, Ottawa, ON.*

Let  $A$ ,  $B$ ,  $C$  be the centres of the three interior circles, let  $D$ ,  $E$ ,  $F$  be the points at which these three circles touch, and let  $O$  be the centre of the large circle as shown.

We denote the radius of the large circle by  $r$ . Note that  $r = AO + 1$ . Therefore, our primary intention is to determine the length of  $AO$ .

By Pythagoras' Theorem, we have

$$AD = \sqrt{AC^2 - DC^2} = \sqrt{4 - 1} = \sqrt{3}.$$

Furthermore, since  $\triangle AEO \sim \triangle ADC$ , we get  $\frac{AO}{AE} = \frac{AC}{AD}$ ; that is,

$$AO = \frac{AC}{AD} \times AE = \frac{2}{\sqrt{3}}.$$

Hence,  $r = \frac{2}{\sqrt{3}} + 1$ . Now the area of the large circle is

$$\pi r^2 = \pi \left(\frac{2}{\sqrt{3}} + 1\right)^2 = \pi \left(\frac{7 + 4\sqrt{3}}{3}\right).$$

*One incorrect solution was received.*

## Relay

**R1.** The sum of five consecutive numbers is 130. Call the smallest of these numbers  $A$ .

Write the value of  $A$  in Box #1 of the Relay Answer Sheet.

*Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

Let the five consecutive numbers be  $x - 2$ ,  $x - 1$ ,  $x$ ,  $x + 1$ ,  $x + 2$ . Their sum is  $5x = 130$ , implying that  $x = 26$ . The smallest is  $A = x - 2 = 24$ .

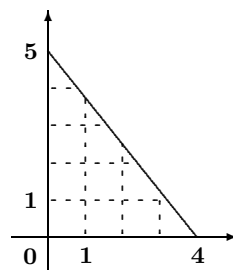
**R2.** A triangle has vertices at  $(0, 0)$ ,  $(A/6, 0)$ , and  $(0, 5)$ . How many points with integer coordinates lie inside the triangle?

Write your answer,  $B$ , in Box #2 of the Relay Answer Sheet.

*Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

The number of lattice points on the boundary of the triangle is clearly 10, since the hypotenuse does not intersect any lattice points other than its end-points. Let  $E$  denote the number of lattice points on the sides of the triangle. The area of the triangle is  $\Delta = 10$ . By Pick's Theorem, the number of lattice points inside the triangle is

$$\Delta - E/2 + 1 = 10 - 5 + 1 = 6.$$



[*Ed.*: With an accurate diagram such as the one above, we can simply count the number of lattice points in the interior.]

**R3.** Determine  $c$  if

$$\begin{aligned}(B + 18)c + 7d &= 4, \\ d + e &= 20, \\ e + f &= 36, \\ f + 5c &= 15.\end{aligned}$$

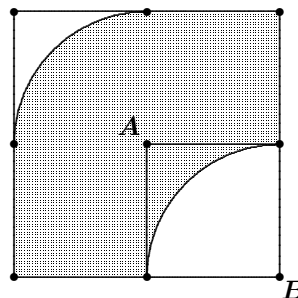
Write the value of  $c$  in Box #3 of the Relay Answer Sheet.

*Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

Working backwards, we have  $f = 15 - 5c = 36 - e$ , which implies that  $5c + 21 = e = 20 - d$ , further implying that  $d = -(5c + 1)$ . Thus, the first equation becomes  $24c - 7(5c + 1) = 4$ ; whence,  $-11c = 11$ , or  $c = -1$ .

**R4.** The sides of the large square in the diagram are twice the length of the sides of the small square. The two arcs are portions of circles with radii equal to the length of the sides of the small square and with centres at the points  $A$  and  $B$ . If the area of the hatched region is  $c^2$ , determine the length of the sides of the small square.

Write this value in Box #4 of the Relay Answer Sheet and hand it to your proctor.



*Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

Split the large square into 4 small squares. Notice that the area missing in the top-left and bottom-right squares sum to one small square. Thus, we have 3 small squares as the shaded area. Suppose the sidelength of the small square is  $x$ . Then  $3x^2 = c^2 = 1$ , yielding  $x = \frac{1}{\sqrt{3}}$ .

That brings us to the end of another issue.