Pólya's Paragon

What's the difference? (Part 2)

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When last we met, in the March issue [2004 : 77], we began our analysis of sequences by looking at the sequences of differences, second differences, and so on. Let's search for a relationship between the differences and the original sequence.

Consider the sequence $\{t_n\}$. In March, we numbered the terms in our sequences starting with $n = 1$, but now we would like to start with $n = 0$ (since this will make our formulas look nicer). We recall the notation $\{{}_1d_n\}$ for the sequence of first differences, $\{{}_2d_n\}$ for the sequence of second differences, and so on.

Try to extend the following table of differences until a pattern emerges. Do **not** read on until you see the pattern.

t_n	${}_1d_n$	${}_2d_n$	$\dots$
t_0			
	$-t_0 + t_1$		
t_1		$t_0 - 2t_1 + t_2$	
	$-t_1 + t_2$		$\dots$
t_2		$t_1 - 2t_2 + t_3$	
	$-t_2 + t_3$		$\dots$
t_3		$t_2 - 2t_3 + t_4$	
	$-t_3 + t_4$		
t_4			
$\vdots$	$\vdots$	$\vdots$	

If you have extended the table to the 5^{th} differences (try it now if you haven't already!), you should have found that those entries are

$$\begin{gathered} -t_0 + 5t_1 - 10t_2 + 10t_3 - 5t_4 + t_5 \\ -t_1 + 5t_2 - 10t_3 + 10t_4 - 5t_5 + t_6 \\ \vdots \end{gathered}$$

Each n^{th} difference seems to be an alternating sum of multiples of $n+1$ consecutive terms from the original sequence. The multiples are just the binomial coefficients. Thus,

$${}_kd_n \;=\; \sum_{i=0}^{k} (-1)^i \binom{k}{i}\, t_{n+k-i}\,.$$

We can use the binomial inversion formula given by Hathout in the September 2003 issue [2003 : 275] to get

$$t_{n+k} \;=\; \sum_{i=0}^{n} \binom{n}{i}\, {}_{i}d_{m}\,.$$

If the given sequence $\{t_n\}$ is really a polynomial in disguise, then at some point the differences will all be zero (as we saw last time), giving us

$$t_n \;=\; \sum_{i=0}^{n} \binom{n}{i}\, {}_{i}d_{0} \;=\; \sum_{i=0}^{r} \binom{n}{i}\, {}_{i}d_{0}\,,$$

where r is the degree of the polynomial. Note that the numbers ${}_{i}d_{0}$ in this formula are the top entries in the columns of the table of differences.

Let's return to the problem that we started with back in March:

"Little Johnnie encounters the following list of numbers 12, 49, 62, 57, 40, 17, $\ldots$. *What is the next number in the list?"*

Constructing a difference table for this problem yields:

t_n	${}_{1}d_n$	${}_{2}d_n$	${}_{3}d_n$	${}_{4}d_n$
12				
	37			
49		−24		
	13		6	
62		−18		0
	−5		6	
57		−12		0
	−17		6	
40		−6		
	−23			
17				

We see that the original sequence was a cubic relation ($r = 3$) and that

$$\begin{aligned}
t_n &= \sum_{i=0}^{r} \binom{n}{i}\, {}_{i}d_{0} \\
&= 12\binom{n}{0} + 37\binom{n}{1} - 24\binom{n}{2} + 6\binom{n}{3} \\
&= 12 + 37n - 24\cdot\frac{n(n-1)}{2} + 6\cdot\frac{n(n-1)(n-2)}{6} \\
&= 12 + 37n - 12n^2 + 12n + n^3 - 3n^2 + 2n \\
&= n^3 - 15n^2 + 51n + 12\,.
\end{aligned}$$

Hence, we see that the next number in Little Johnny's list is $t_6 = -6$.

We can use these relationships also in cases where the sequence of n^{th} differences is known but not just constant like in the polynomial case. We will examine some such cases (including the homework from the March issue) in our next installment.