MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Larry Rice (University of Waterloo) and Dan MacKinnon (Ottawa Carleton District School Board).

Mayhem Problems

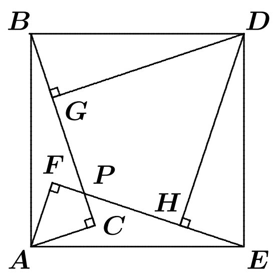
Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier novembre 2004**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M144. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Sur l'hypoténuse AB d'un triangle rectangle ABC on dessine un carré $ABDE$ de manière que C en soit un point intérieur. On dessine ensuite un triangle rectangle directement semblable BDG de sorte que G soit aussi un point intérieur du carré. On dessine finalement deux triangles rectangles indirectement semblables EDH et AEF, tels que H et F soient des points intérieurs du carré. Soit P le point d'intersection de BC et EF. Déterminer l'aire du quadrilatère $DGPH$ en fonction des côtés CA et CB du triangle rectangle original.



M145. *Proposé par Ovidiu-Gabriel Dinu, Balcesti-Valcea, Roumanie.*

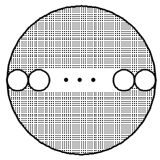
Trouver tous les nombres naturels n pour lesquels n, $n+2$, $n+6$, $n+8$ et $n+14$ sont premiers.

M146. *Proposé par Mohammed Aassila, Strasbourg, France.*

Soit a, b et c trois nombres positifs satisfaisant $a+b+c=1$. Montrer que

$$(ab)^{5/4}+(bc)^{5/4}+(ca)^{5/4} \;<\; \frac{1}{4}\,.$$

M147. *Proposé par l'Equipe de Mayhem.*



Le diamètre d'un grand cercle est divisé en n parties égales pour construire n cercles plus petits, comme dans la figure. Trouver n, sachant que le rapport entre l'aire hachurée et l'aire non hachurée du grand cercle est $3:1$.

M148. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Soit $x>1$, $y>1$, $z>1$ et $x^2=yz$. Trouver la valeur de

$$\left(\log_{zx} xy^4z\right)\left(\log_{xy} xyz^4\right)\,.$$

M149. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Soit ABC un triangle rectangle de Héron possédant la propriété suivante : son aire est λ fois son périmètre, avec λ un entier positif. Trouver toutes les solutions (a,b,λ). (Un triangle de Héron est un triangle dont la longueur des côtés et l'aire sont des nombres entiers.)

M150. *Proposé par Arkady Alt, San Jose, CA, USA.*

Soit deux nombres complexes z_1 et z_2 satisfaisant les conditions

$$\begin{aligned} z_1+z_2 &= -(i+1)\,, \\ z_1\cdot z_2 &= -i\,. \end{aligned}$$

Sans calculer z_1 et z_2, trouver $z_1\cdot\overline{z_2}$.

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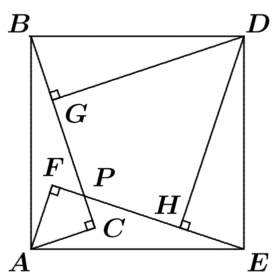
M144. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*



A square $ABDE$ is drawn on the hypotenuse AB of right triangle ABC so that C lies in the interior of the square. A directly similar right triangle BDG is drawn so that G lies in the interior of the square. Indirectly similar right triangles EDH and AEF are drawn so that H and F lie in the interior of the square. Let BC and EF intersect at P. Determine the area of quadrilateral $DGPH$ in terms of the legs CA and CB of the original right triangle.

M145. *Proposed by Ovidiu-Gabriel Dinu, Balcesti-Valcea, Romania.*

Find all natural numbers n for which n, $n+2$, $n+6$, $n+8$, and $n+14$ are prime numbers.

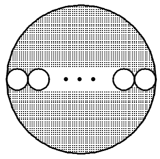
M146. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let a, b, c be three positive numbers satisfying $a+b+c=1$. Prove that

$$(ab)^{5/4}+(bc)^{5/4}+(ca)^{5/4} \;<\; \frac{1}{4}\,.$$

M147. *Proposed by the Mayhem staff.*

The diameter of a large circle is broken into n equal parts to construct n smaller circles, as shown. Determine n so that the ratio of the shaded area to the unshaded area in the large circle is $3:1$.



M148. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let $x>1$, $y>1$, $z>1$ and $x^2=yz$. Determine the value of

$$\left(\log_{zx} xy^4z\right)\left(\log_{xy} xyz^4\right)\;.$$

M149. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A right-angled Heron triangle ABC has the following property: the area is λ times the perimeter, where λ is a positive integer. Determine all solutions (a,b,λ). (A Heron triangle is a triangle with integer sides and integer area.)

M150. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let two complex numbers z_1 and z_2 satisfy the conditions

$$\begin{aligned} z_1+z_2 &= -(i+1)\,, \\ z_1\cdot z_2 &= -i\,. \end{aligned}$$

Without calculating z_1 and z_2, find $z_1\cdot\overline{z_2}$.

Pólya's Paragon

What's the difference? (Part 2)

Shawn Godin

When last we met, in the March issue [2004 : 77], we began our analysis of sequences by looking at the sequences of differences, second differences, and so on. Let's search for a relationship between the differences and the original sequence.

Consider the sequence $\{t_n\}$. In March, we numbered the terms in our sequences starting with $n = 1$, but now we would like to start with $n = 0$ (since this will make our formulas look nicer). We recall the notation $\{{}_1d_n\}$ for the sequence of first differences, $\{{}_2d_n\}$ for the sequence of second differences, and so on.

Try to extend the following table of differences until a pattern emerges. Do **not** read on until you see the pattern.

t_n	${}_1d_n$	${}_2d_n$	$\dots$
t_0			
	$-t_0 + t_1$		
t_1		$t_0 - 2t_1 + t_2$	
	$-t_1 + t_2$		$\dots$
t_2		$t_1 - 2t_2 + t_3$	
	$-t_2 + t_3$		$\dots$
t_3		$t_2 - 2t_3 + t_4$	
	$-t_3 + t_4$		
t_4			
$\vdots$	$\vdots$	$\vdots$	

If you have extended the table to the 5th differences (try it now if you haven't already!), you should have found that those entries are

$$-t_0 + 5t_1 - 10t_2 + 10t_3 - 5t_4 + t_5$$
$$-t_1 + 5t_2 - 10t_3 + 10t_4 - 5t_5 + t_6$$
$$\vdots$$

Each n^{th} difference seems to be an alternating sum of multiples of $n+1$ consecutive terms from the original sequence. The multiples are just the binomial coefficients. Thus,

$${}_kd_n \;=\; \sum_{i=0}^{k}(-1)^i\binom{k}{i}\,t_{n+k-i}\,.$$

We can use the binomial inversion formula given by Hathout in the September 2003 issue [2003 : 275] to get

$$t_{n+k} \;=\; \sum_{i=0}^{n} \binom{n}{i}\, {}_i d_m \,.$$

If the given sequence $\{t_n\}$ is really a polynomial in disguise, then at some point the differences will all be zero (as we saw last time), giving us

$$t_n \;=\; \sum_{i=0}^{n} \binom{n}{i}\, {}_i d_0 \;=\; \sum_{i=0}^{r} \binom{n}{i}\, {}_i d_0 \,,$$

where r is the degree of the polynomial. Note that the numbers ${}_i d_0$ in this formula are the top entries in the columns of the table of differences.

Let's return to the problem that we started with back in March:

"Little Johnnie encounters the following list of numbers 12, 49, 62, 57, 40, 17, $\ldots$. *What is the next number in the list?"*

Constructing a difference table for this problem yields:

t_n	${}_1d_n$	${}_2d_n$	${}_3d_n$	${}_4d_n$
12				
	37			
49		−24		
	13		6	
62		−18		0
	−5		6	
57		−12		0
	−17		6	
40		−6		
	−23			
17				

We see that the original sequence was a cubic relation ($r = 3$) and that

$$\begin{aligned}
t_n &= \sum_{i=0}^{r} \binom{n}{i}\, {}_i d_0 \\
&= 12\binom{n}{0} + 37\binom{n}{1} - 24\binom{n}{2} + 6\binom{n}{3} \\
&= 12 + 37n - 24 \cdot \frac{n(n-1)}{2} + 6 \cdot \frac{n(n-1)(n-2)}{6} \\
&= 12 + 37n - 12n^2 + 12n + n^3 - 3n^2 + 2n \\
&= n^3 - 15n^2 + 51n + 12 \,.
\end{aligned}$$

Hence, we see that the next number in Little Johnny's list is $t_6 = -6$.

We can use these relationships also in cases where the sequence of n^{th} differences is known but not just constant like in the polynomial case. We will examine some such cases (including the homework from the March issue) in our next installment.