SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2548★. [2000 : 238] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana, USA.*

Let $a(1) = 1$ and, for $n \geq 2$, define $a(n) = \lfloor a(n-1)/2 \rfloor$, if this is not in $\{0, a(1), \ldots, a(n-1)\}$, and $a(n) = 3a(n-1)$ otherwise.

(a) Does any positive integer occur more than once in this sequence?

(b) Does every positive integer occur in this sequence?

Solution by Mateusz Kwaśnicki, student, Wroclaw University of Technology, Poland.

Every positive integer occurs exactly once in the sequence. We will prove that, more generally, when the numbers 2 and 3 in the definition of the sequence are replaced by arbitrary integers p and q greater than 1, then, in the sequence,

(a) (Uniqueness) No positive integer occurs more than once, and

(b) (Existence) Every positive integer occurs if and only if the following condition is satisfied: $\log_q p$ is irrational, or equivalently,

$$p^n = q^m \text{ for integers } m,\, n \quad \Longrightarrow \quad m = n = 0\,.$$

Note that the condition that $\log_q p$ is irrational is satisfied, in particular, when p and q are relatively prime.

Let p, q be any integers greater that 1. Define the *multiplying-dividing (M-D) sequence* (a_n) for the multiplier p and divisor q recursively as follows. Begin with $a_1 = 1$, and take a_{n+1} equal to $\lfloor a_n/q \rfloor$ if this number is positive and has not yet occured in the sequence, or equal to pa_n otherwise.

For instance, if $p = 3$ and $q = 2$ (the original problem), then the corresponding M-D sequence is 1, 3, 9, 4, 2, 6, 18, 54, 27, 13, 39, 19, 57, 28, 14, 7, 21, $\ldots$.

As usual, let $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{R}$ denote the sets of integers, positive integers, and real numbers, respectively. We define

$$A_n = \{0\} \cup \{a_m : m \leq n\} \quad \text{and} \quad A = \{a_m : m \in \mathbb{Z}^+\}\,.$$

Then the definition of the M-D sequence (a_n) may be expressed as follows:

$$\begin{aligned} a_1 &= 1\,, \\ a_{n+1} &= \begin{cases} pa_n & \text{if } \lfloor a_n/q \rfloor \in A_n\,, \\ \lfloor a_n/q \rfloor & \text{if } \lfloor a_n/q \rfloor \notin A_n\,. \end{cases} \end{aligned}$$

Lemma 1. Let $x \in \mathbb{R}$, $u \in \mathbb{Z}$, and $p \in \mathbb{Z}^+$. If $pu = \lfloor px \rfloor$, then $u = \lfloor x \rfloor$.

Proof. If $pu = \lfloor px \rfloor$, then

$$u = \lfloor u \rfloor = \left\lfloor \frac{pu}{p} \right\rfloor = \left\lfloor \frac{\lfloor px \rfloor}{p} \right\rfloor = \left\lfloor \frac{px}{p} \right\rfloor = \lfloor x \rfloor .$$

□

Lemma 2. For any $n \in \mathbb{Z}^+$, if $\lfloor a_n/q \rfloor \in A_n$, then $\lfloor a_n/q^k \rfloor \in A_n$, for all $k \in \mathbb{Z}^+$.

Proof. Fix $n \in \mathbb{Z}^+$ such that $\lfloor a_n/q \rfloor \in A_n$. We apply induction. Assume that $\lfloor a_n/q^k \rfloor \in A_n$ for some $k \in \mathbb{Z}^+$. If $\lfloor a_n/q^k \rfloor = 0$, then we have $\lfloor a_n/q^{k+1} \rfloor = 0 \in A_n$. If $\lfloor a_n/q^k \rfloor \neq 0$, then $\lfloor a_n/q^k \rfloor = a_m$ for some $m \leq n$. Note that m cannot equal n, since $\lfloor a_n/q^k \rfloor < a_n$; thus, $m < n$. It follows that either $\lfloor a_m/q \rfloor \in A_m \subset A_n$ or $\lfloor a_m/q \rfloor = a_{m+1} \in A_n$. Hence, $\lfloor a_n/q^{k+1} \rfloor = \lfloor a_m/q \rfloor \in A_n$, which completes the induction. □

Proof of Uniqueness. Assume, on the contrary, that $a_i = a_j$ for some $i \neq j$. Let n be the smallest integer such that $a_n = a_m$ for some m, $0 < m < n$. Then $a_n \in A_{n-1}$. Hence, $a_n \neq \lfloor a_{n-1}/q \rfloor$, and so $a_n = pa_{n-1}$. Let $i \leq m$ be such that

$$a_m = \left\lfloor \frac{a_{m-1}}{q} \right\rfloor, \quad a_{m-1} = \left\lfloor \frac{a_{m-2}}{q} \right\rfloor, \quad \ldots , \quad a_{i+1} = \left\lfloor \frac{a_i}{q} \right\rfloor, \quad a_i = pa_{i-1} .$$

(Note that such i exists, since $a_2 = pa_1$.) Let $k = m - i$. Then

$$pa_{n-1} = a_n = a_m = \left\lfloor \frac{a_i}{q^k} \right\rfloor = \left\lfloor \frac{pa_{i-1}}{q^k} \right\rfloor .$$

Hence, $a_{n-1} = \lfloor a_{i-1}/q^k \rfloor$, using Lemma 1. Now, since $a_i = pa_{i-1}$, we have $\lfloor a_{i-1}/q \rfloor \in A_{i-1}$. By Lemma 2, it follows that $a_{n-1} = \lfloor a_{i-1}/q^k \rfloor \in A_{i-1}$. Then $a_{n-1} = a_j$ for some $j \leq i - 1 < n - 1$, contradicting our choice of n. Hence, our assumption was false, and thus we have proved uniqueness. □

Proof of Existence – Part I. We will prove the necessity of the condition that $\log_q p$ is irrational. Suppose that $\log_q p$ is rational, say $\log_q p = m/n$, for some relatively prime integers m, $n > 0$. Then $q^m = p^n$. Hence, $q^{1/n} = p^{1/m}$ is an integer greater than 1; denote it by d. Induction shows that $a_n = d^{k_n}$ for some non-negative integer k_n. In particular, there are positive integers which do not occur in the sequence (a_n). □

To prove the sufficiency of the condition that $\log_q p$ is irrational, we will need to probe a little deeper, using some ideas from topological dynamics. The quotient group of the additive (topological) groups $\mathbb{R}$ and $\mathbb{Z}$ (which is isomorphic to $[0, 1)$ with addition 'modulo 1') will be denoted by $\mathbb{T}$. Elements of $\mathbb{T}$ are cosets $[a] = a + \mathbb{Z} = \{a + n : n \in \mathbb{Z}\}$, where $a \in \mathbb{R}$. Let $\kappa : \mathbb{R} \to \mathbb{T}$ be the canonical homomorphism defined by $\kappa(a) = [a]$. We recall that a subset U of $\mathbb{T}$ is open if and only if $\kappa^{-1}(U)$ is an open subset of $\mathbb{R}$ (with the usual topology).

Lemma 3. The canonical homomorphism $\kappa : \mathbb{R} \to \mathbb{T}$ is an open mapping.

Proof. Let V be an open subset of $\mathbb{R}$, and define $U = \kappa(V)$. Then

$$\begin{aligned}\kappa^{-1}(U) = \{a : [a] \in U\} &= \{a : \exists\, v \in V, a \in [v]\}\\ &= \{a : \exists\, v \in V, \exists\, n \in \mathbb{Z}, a = v + n\}\\ &= \{a : \exists\, n \in \mathbb{Z}, a \in V + n\} = \bigcup_{n\in\mathbb{Z}} (V + n)\,,\end{aligned}$$

where $V + n = \{v + n : v \in V\}$. This implies that $\kappa^{-1}(U)$ is open, since $V + n$ is open for every $n \in \mathbb{Z}$. Hence, U is open, which proves that κ is an open mapping. □

Lemma 4. If $b \in \mathbb{Z}^+\backslash A$, then $q^k b + l \in \mathbb{Z}^+\backslash A$ for any $k \in \mathbb{Z}^+$ and $l \in \mathbb{Z}$ such that $0 \leq l < q^k$.

Proof. Let $a \in A$. Then $a = a_n$ for some n. Either $\lfloor a_n/q \rfloor = a_{n+1} \in A$ or $\lfloor a_n/q \rfloor \in A_n \subset A \cup \{0\}$. Hence, $\lfloor a/q \rfloor \in A$ or $\lfloor a/q \rfloor = 0$. By induction, we get $\lfloor a/q^k \rfloor \in A$ or $\lfloor a/q^k \rfloor = 0$. Writing $b = \lfloor a/q^k \rfloor$, we get the desired result. □

Lemma 5 (Topological Dynamics Lemma). Let a be an irrational number, and let $U \subset \mathbb{T}$ be a nonempty open set. Then there exists $N \in \mathbb{Z}^+$ such that, for every $x \in \mathbb{R}$, there exists n, $0 \leq n \leq N$, such that $[x + na] \in U$.

Proof. Fix $x \in \mathbb{R}$, and let $A = \{[x + na] : n \geq 0\}$. Then A is dense in $\mathbb{T}$. (Since the proof of this statement is well known and long, we omit it.) Hence, $[x + na] \in U$ for some n. Therefore, $[x] \in \bigcup\limits_{n=0}^{\infty} (U - [na])$. Since x was arbitrary, $\bigcup\limits_{n=0}^{\infty} (U - [na]) = \mathbb{T}$. By the compactness of $\mathbb{T}$, there exists $N \in \mathbb{Z}^+$ such that $\bigcup\limits_{n=0}^{N} (U - [na]) = \mathbb{T}$.

Take any $x \in \mathbb{R}$. Then $[x] \in U - [na]$ for some n, $0 \leq n \leq N$. For this n, we have $[x + na] \in U$. □

Proof of Existence – Part II. Assume that $\log_q p$ is irrational. We intend to show that $\mathbb{Z}^+\backslash A = \emptyset$. Assume, to the contrary, that $\mathbb{Z}^+\backslash A$ is not empty, and let a be any element of this set. We will show that $q^t a + s \in A$ for some t, $s \in \mathbb{Z}^+$, $0 \leq s < q^t$, which contradicts Lemma 4.

The main idea is that the difference between $\log_q a$ and $\log_q(q^t a + s)$ is nearly an integer. This means that the distance between $[\log_q a]$ and $[\log_q(q^t a + s)]$ in $\mathbb{T}$ is small. Hence, it is sufficient to show that the set of cosets $[\log_q a_n]$ cannot be separated from $[\log_q a]$ for all n. It will turn out that the sequence of $[\log_q a_n]$ has much in common with rotations, well studied transformations on $\mathbb{T}$.

The uniqueness proved above implies that $\lim\limits_{n\to\infty} a_n = \infty$. It follows that there are infinitely many n such that $a_{n+1} = pa_n$; let k_1, k_2, ... be the increasing sequence of all those n. Thus,

$$a_{k_{n+1}} \;=\; \left\lfloor \frac{a_{k_n+1}}{q^{k_{n+1}-k_n-1}} \right\rfloor, \qquad a_{k_n+1} \;=\; pa_{k_n}\,. \tag{1}$$

Let $b_n = \log_q a_n$ and $\alpha = \log_q p$ (which, we recall, is irrational). Define

$$\epsilon_n \;=\; \log_q \frac{a_{k_n+1}}{q^{k_{n+1}-k_n-1}} - \log_q a_{k_{n+1}} \;=\; b_{k_n+1} - b_{k_{n+1}} - k_{n+1} + k_n + 1\,.$$

Using (1), and noting that $\lfloor x\rfloor \le x < \lfloor x\rfloor + 1$ for $x \in \mathbb{R}$, we obtain $\epsilon_n \ge 0$ and

$$\epsilon_n \;<\; \log_q(a_{k_{n+1}}+1) - \log_q a_{k_{n+1}} \;=\; \log_q\left(1 + \frac{1}{a_{k_{n+1}}}\right) \;<\; \frac{1}{(\ln q)a_{k_{n+1}}}\,.$$

Thus,

$$0 \;\le\; \epsilon_n \;<\; \frac{1}{(\ln q)a_{k_{n+1}}}\,. \tag{2}$$

By (1) and the definition of ϵ_n,

$$\begin{aligned} [b_{k_{n+1}}] \;&=\; [b_{k_n+1} - \epsilon_n]\,, \\ [b_{k_n+1}] \;&=\; [\log_q(pa_{k_n})] = [b_{k_n} + \alpha]\,. \end{aligned} \tag{3}$$

This is what we needed: $[b_{k_{n+1}}]$ is very close to $[b_{k_n+1}]$, which is an irrational rotation of $[b_{k_n}]$. Hence, we can apply methods of topological dynamics.

Before we proceed with some technical details, let us recall that we are looking for a_m such that $a_m = q^t a + s$ for some $t,\ s \in \mathbb{Z}^+$, $0 \le s < q^t$, which is equivalent to $\log_q a_m \in \big(\log_q a + t\,,\ \log_q(a+1) + t\big)$.

Define

$$\delta \;=\; \log_q(a+\tfrac{1}{2}) - \log_q a\,, \quad V \;=\; \big(\log_q(a+\tfrac{1}{2}), \log_q(a+1)\big), \quad U \;=\; \kappa(V)\,.$$

Since U is a nonempty open subset of $\mathbb{T}$ and α is irrational, Lemma 5 implies that there exists $L \in \mathbb{Z}^+$ such that for every $x \in \mathbb{R}$ we have $[x + l\alpha] \in U$ for some l, $0 \le l \le L$.

Let M be large enough so that $L < M\delta \ln q$. Since $\lim_{n\to\infty} a_n = \infty$, there exists N such that $a_n > M$ for $n \ge N$. Fix any n such that $k_n \ge N$. For some l, $0 \le l \le L$, we have $[b_{k_n} + l\alpha] \in U$. Equivalently, for some $i \in \mathbb{Z}$,

$$b_{k_n} + l\alpha + i \in V\,.$$

By definition of V,

$$\log_q(a + \tfrac{1}{2}) \;<\; b_{k_n} + l\alpha + i \;<\; \log_q(a+1) \tag{4}$$

Let $m = n + l$. Using (3) and a simple induction argument, we get

$$[b_{k_m}] \;=\; [b_{k_n} \;+\; l\alpha \;-\; \epsilon_n \;-\; \epsilon_{n+1} \;-\; \ldots \;-\; \epsilon_{m-1}]\,.$$

Again this is equivalent to

$$b_{k_m} = b_{k_n} + l\alpha - \epsilon_n - \epsilon_{n+1} - \ldots - \epsilon_{m-1} + j$$

for some $j \in \mathbb{Z}$. Recall that if $n \le \nu < m$, then $k_{\nu+1} > k_n \ge N$ and so $a_{k_{\nu+1}} > M$. Then, according to (2), we have $0 \le \epsilon_\nu < (M\ln q)^{-1}$, so that

$$b_{k_n} + l\alpha + j - \frac{m-n}{M\ln q} < b_{k_m} \le b_{k_n} + l\alpha + j\,.$$

But M was defined so that

$$\frac{m-n}{M\ln q} = \frac{l}{M\ln q} \le \frac{L}{M\ln q} < \delta\,.$$

Hence,

$$b_{k_n} + l\alpha + j - \delta < b_{k_m} \le b_{k_n} + l\alpha + j\,.$$

Together with (4) and the definition of δ, this leads us to

$$\log_q(a+\tfrac{1}{2}) + j - i - \left(\log_q(a+\tfrac{1}{2}) - \log_q a\right) < b_{k_m} < \log_q(a+1) + j - i\,.$$

Finally, we get

$$\log_q a + j - i < b_{k_m} < \log_q(a+1) + j - i\,.$$

In terms of a_ν, this means $q^{j-i}a < a_{k_m} < q^{j-i}(a+1)$. It follows that $j-i > 0$ and $a_{k_m} = q^t a + s$ for $t = i-j$ and some s such that $0 < s < q^t$, which contradicts Lemma 4.

This proves that $\mathbb{Z}^+ \backslash A$ is empty, or in other words, that every positive integer occurs in the sequence. □

2799★. [2002 : 536, 2003 : 47, 2003 :531–534] *Proposed Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove or disprove the inequality

$$\sum_{\substack{i,\,j\,\in\,\{1,\,2,\,\ldots\,,n\}\\ 1\le i<j\le n}} \frac{1}{1-x_ix_j} \le \binom{n}{2}\frac{1}{1-\frac{1}{n^2}}\,,$$

where $\sum\limits_{j=1}^{n} x_j = 1$, $x_j \ge 0$.

[*Ed:* The solution to this problem, originally published in [2003 : 531–534], was incorrect. The error was identified by Vasile Cîrtoaje. Below is his generalization of the problem and its solution.]

Generalization and solution by Vasile Cîrtoaje, University of Ploiesti, Romania.

We prove a slightly more general result:

Theorem. If $0 < s \leq 1$ and $x_1, x_2, \ldots, x_n \geq 0$ are real numbers such that $x_1 + x_2 + \cdots + x_n = s$, then

$$\sum_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \leq \binom{n}{2} \frac{n^2}{n^2 - s^2}.$$

Proof. Since the inequality is clearly true for $n = 2$, we will assume $n \geq 3$. Also, we suppose (without loss of generality) that $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$. The set $\{(x_1, x_2, \ldots, x_n) : x_1 + x_2 + \cdots + x_n = s\}$ is a compact set in $\mathbb{R}^n$, and the function

$$F(x_1, x_2, \cdots, x_n) = \sum_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}$$

is continuous on this set. Consequently, F attains a maximum value at one or more points of the set. We will prove, in two steps, that the maximum occurs only at the point $\left(\frac{s}{n}, \frac{s}{n}, \cdots, \frac{s}{n}\right)$, whence the desired inequality follows.

In the first step, we will show that for $s \leq \sqrt{3}$, if the function F is maximal at $(x_1, x_2, \ldots, x_n)$, then $x_1 = x_2$.

In the second step, we will show that if the function F is maximal at $(x_1, x_2, \ldots, x_n)$ with $x_1 = x_2 = \cdots = x_i$, where $i \in \{2, 3, \ldots, n-1\}$, then $x_{i+1} = x_i$. These statements will complete the proof.

Step 1.

To prove the claimed statement, we suppose that $x_1 > x_2$ and show that F is not maximal at $(x_1, x_2, \ldots, x_n)$. More precisely, we show that

$$F(x_1, x_2, \ldots, x_n) < F\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n\right).$$

For convenience, we replace x_1, x_2, and $\frac{x_1 + x_2}{2}$ by x, y, and t, respectively. Thus, the inequality can be written in the form

$$\sum_{j=3}^{n} \left(\frac{1}{1 - x x_j} + \frac{1}{1 - y x_j} - \frac{2}{1 - t x_j}\right) < \frac{1}{1 - t^2} - \frac{1}{1 - xy}.$$

After combining terms and dividing by the positive factor $t^2 - xy$, we get

$$\sum_{j=3}^{n} \left(\frac{2x_j^2}{(1 - x x_j)(1 - y x_j)(1 - t x_j)}\right) < \frac{1}{(1 - t^2)(1 - xy)}. \tag{1}$$

The condition $s \le \sqrt{3}$ is necessary. Indeed, letting $x_4 = \cdots = x_n = 0$ and $x = y = x_3 = \frac{s}{3}$, we get $x < \frac{1}{\sqrt{3}}$ from (1), which implies $s < \sqrt{3}$.

Now, we notice that

$$\sum_{j=3}^{n} \left(\frac{2x_j^2}{(1-xx_j)(1-yx_j)(1-tx_j)} \right) \le \frac{2(x_3^2+\cdots+x_n^2)}{(1-xx_3)(1-yx_3)(1-tx_3)} \le \frac{2(x_3^2+\cdots+x_n^2)}{(1-xy)(1-y^2)(1-t^2)}\,.$$

Thus, to prove (1), it suffices to show that

$$2(x_3^2+\cdots+x_n^2)+y^2 \;<\; 1\,. \tag{2}$$

We introduce the notation $r = x_3 + \cdots + x_n$, and consider two cases for r.

Case 1. $r \le y$. Since $x_3^2 + \cdots + x_n^2 \le r^2$, it is sufficient to show that

$$2r^2 + y^2 \;<\; 1\,.$$

From $x > y \ge r$ and $x + y + r = s$, we obtain $r < \frac{s}{3}$ and $2y + r < s$. Thus,

$$\begin{aligned} 2r^2+y^2-1 &< 2r^2 + \left(\frac{s-r}{2}\right)^2 - 1 \\ &= \frac{1}{4}\left(r - \frac{s}{3}\right)(s+9r) + \left(\frac{s^2}{3} - 1\right) \;<\; 0\,. \end{aligned}$$

Case 2. $r > y$. The condition $r > y$ implies $n \ge 4$, (since in the case $n = 3$ we have $r = x^3 \le y$) and $y > 0$ (since $y = 0$ implies $r = 0$). From the order relation $x > y \ge x_3 \ge \cdots \ge x_n$, we get $y < r \le (n-2)y$. Thus, we may conclude there is $k \in \{1, 2, \ldots, n-3\}$ such that

$$ky \;<\; r \;\le\; (k+1)y\,. \tag{3}$$

According to the Majorization Inequality (of Karamata) applied to the convex function $f(x) = x^2$, the following inequality holds

$$x_3^2 + \cdots + x_n^2 \;\le\; ky^2 + (r-ky)^2 + (n-k-3)\cdot 0^2\,.$$

Consequently, to prove (2) it suffices to show that

$$(2k+1)y^2 + 2(r-ky)^2 \;<\; 1\,. \tag{4}$$

[*Ed:* Karamata's Majorization Theorem is as follows. If two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ having n components, a_i and b_i, $i = 1, 2, \ldots, n$, arranged in non-increasing order of magnitude, satisfy

$$\sum_{i=1}^{k} a_i \;\ge\; \sum_{i=1}^{k} b_i\,, \qquad k = 1, 2, \ldots, n-1,$$

$$\text{and} \qquad \sum_{i=1}^{n} a_i \;=\; \sum_{i=1}^{n} b_i\,,$$

we say that $\overrightarrow{A}$ *majorizes* $\overrightarrow{B}$, which we denote as $\overrightarrow{A} \succ \overrightarrow{B}$. For a convex function $F(x)$, we then have

$$F(a_1) + F(a_2) + \cdots + F(a_n) \;\geq\; F(b_1) + F(b_2) + \cdots + F(b_n)\,.$$

[1] Klamkin, Murray S., On a "Problem of the Month", [2002 : 86–87].

[2] Klamkin, Murray S., Quickie Inequalities, π *in the sky*, September 2003, 26–29.

[3] Marshall, A. M. and Olkin, I., *Inequalities: Theory of Majorization and Its Applications,* Academic Press, New York, 1979.]

Letting $x = (p - k - 1)y$ and $y = qs$, we have

$$r \;=\; s - x - y \;=\; \left(\frac{1}{q} - p + k\right) y\,.$$

Therefore,

$$r - ky \;=\; \left(\frac{1}{q} - p\right) y\,.$$

From this result and (3), we get

$$\frac{1}{p+1} \;\leq\; q \;<\; \frac{1}{p}\,.$$

Also, the condition $x > y$ implies that

$$p \;>\; k + 2\,. \tag{5}$$

The inequality (4) is equivalent to

$$2k + 1 + 2\left(\frac{1}{q} - p\right)^2 \;<\; \frac{1}{y^2}\,.$$

Since $y = qs \leq q\sqrt{3}$, it suffices to prove that

$$3(2k+1)q^2 + 6(1 - pq)^2 \;<\; 1\,. \tag{6}$$

In order to prove (6), we consider the function

$$f(u) \;=\; 3(2k+1)u^2 + 6(1 - pu)^2 - 1\,.$$

We have to show that $f(u) < 0$ for $\frac{1}{p+1} \leq u < \frac{1}{p}$. Since $f(u)$ is a convex function, it suffices to show that $f\left(\frac{1}{p+1}\right) < 0$ and $f\left(\frac{1}{p}\right) \leq 0$. Indeed, according to (5), we have

$$f\left(\frac{1}{p+1}\right) \;=\; \frac{6k+9}{(p+1)^2} - 1 \;<\; \frac{6k+9}{(k+3)^2} - 1 \;=\; \frac{-k^2}{(k+3)^2} \;<\; 0\,,$$

and

$$f\left(\frac{1}{p}\right) \;=\; \frac{3(2k+1)}{p^2} - 1 \;<\; \frac{3(2k+1)}{(k+2)^2} - 1 \;=\; \frac{-(k-1)^2}{(k+2)^2} \;\leq\; 0\,.$$

Step 2.

To prove the claim, we proceed the same way as in Step 1. We consider $x_i > x_{i+1}$ for $2 \le i \le n-1$, and then show that F is not maximal at $(x_1, x_2, \ldots, x_n)$. Using the substitutions $x_i = x$, $x_{i+1} = y$, and $t = \frac{x+y}{2}$, the inequality corresponding to (1) has the form

$$\frac{2(i-1)x^2}{(1-x^2)(1-yx)(1-tx)} + \sum_{j=i+2}^{n} \frac{2x_j^2}{(1-xx_j)(1-yx_j)(1-tx_j)} < \frac{1}{(1-t^2)(1-xy)}\,.$$

Since $x > y \ge x_{i+2} \ge \cdots \ge x_n \ge 0$, it suffices to show that

$$\frac{2(i-1)x^2}{(1-x^2)(1-yx)(1-x^2)} + \frac{\sum_{j=i+2}^{n} 2x_j^2}{(1-x^2)(1-yx)(1-x^2)} < \frac{1}{1-xy}\,.$$

This inequality can be written as

$$2ix^2 + 2(x_{i+2}^2 + \cdots + x_n^2) < 1 + x^4\,. \tag{7}$$

Since $x^4 > 0$, it suffices to show that

$$2ix^2 + 2(x_{i+2}^2 + \cdots + x_n^2) \le 1\,. \tag{8}$$

Case 1. $n = 3$. We have $i = 2$ and $x_{i+2}^2 + \cdots + x_n^2 = 0$. The inequality (8) becomes $4x^2 \le 1$. From $2x + y = s$, we get $x \le \frac{s}{2} \le \frac{1}{2}$, therefore, $4x^2 \le 1$.

Case 2. $n > 3$. Let $r = x_{i+2} + \cdots + x_n$. From $x > y \ge x_{i+2} \ge \cdots \ge x_n \ge 0$ and $ix + y + r = s \le 1$, we obtain

$$4\left(\sum_{j=i+2}^{n} x_j^2\right) \le 4\left(\sum_{j=i+2}^{n} x_j y\right) = 4ry \le (y+r)^2 \le (1-ix)^2\,.$$

Thus, to prove (8), it suffices to show that $4ix^2 + (1-ix)^2 \le 2$. Indeed, we have

$$\begin{aligned} 4ix^2 + (1-ix)^2 - 2 &= (i^2+4i)x^2 - 2ix - 1 \\ &= (ix-1)(3ix+1) - 2i(i-2)x^2 \le 0\,. \end{aligned}$$

This completes the proof. □

Remark. The statement of the theorem is valid for a larger range of s. In our proof, the range is restricted by (8). Really, the inequality (7) allows us to increase the range of s to

$$0 < s \le \sqrt{6} - \sqrt{2} \approx 1.035\,.$$

However, we conjecture the following more general statement.

Conjecture. Let $n \geq 3$, $x_1, x_2, \ldots, x_n \geq 0$, and $s = x_1 + x_2 + \cdots + x_n$.

(a) If $s \leq \sqrt{\dfrac{2n}{n+1}}$, then

$$\sum_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \;\leq\; \frac{n^3(n-1)}{2(n^2 - s^2)}\,;$$

(b) If $\sqrt{\dfrac{2n}{n+1}} \leq s < 2$, then

$$\sum_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \;\leq\; \frac{n(n-1)}{2} + \frac{s^2}{4 - s^2}\,.$$

The inequalities can be written as follows:

(a) $F(x_1, x_2, \ldots, x_n) \;\leq\; F\left(\frac{s}{n}, \frac{s}{n}, \cdots, \frac{s}{n}\right)$;

(b) $F(x_1, x_2, \ldots, x_n) \;\leq\; F\left(\frac{s}{2}, \frac{s}{2}, 0, \cdots, 0\right)$.

2839. [2003 : 239; corrected 2003 : 315] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

Suppose that x, y, and z are real numbers. Prove that

$$\left(x^3 + y^3 + z^3\right)^2 + 3(xyz)^2 \;\geq\; 4\left(y^3z^3 + z^3x^3 + x^3y^3\right)\,.$$

Determine the cases of equality.

I *Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

If $z = 0$, then

$$\begin{aligned}
&(x^3 + y^3 + z^3)^2 + 3(xyz)^2 - 4(y^3z^3 + z^3x^3 + x^3y^3) \\
&\quad = \; (x^3 + y^3)^2 - 4x^3y^3 \; = \; (x^3 - y^3)^2 \; \geq \; 0\,,
\end{aligned}$$

with equality if and only if $x = y$.

If $z \neq 0$, let $u = \frac{x}{z}$ and $v = \frac{y}{z}$. Then (with considerable help from MAPLE) we have

$$\begin{aligned}
&\frac{1}{z^6}\left((x^3 + y^3 + z^3)^2 + 3(xyz)^2 - 4(y^3z^3 + z^3x^3 + x^3y^3)\right) \\
&\quad = \; (u^3 + v^3 + 1)^2 + 3(uv)^2 - 4(u^3v^3 + u^3 + v^3) \\
&\quad = \; (u^2 + u + uv + v^2 + v + 1) \\
&\qquad\quad \cdot (u^4 - u^3v - uv^3 + v^4 - u^3 + u^2v + uv^2 - v^3 + uv - u - v + 1)\,.
\end{aligned}$$

Note that $u^2+u+uv+v^2+v+1=\frac{1}{2}\big((u+v+1)^2+u^2+v^2+1\big)>0$ and

$$\begin{aligned}&u^4-u^3v-uv^3+v^4-u^3+u^2v+uv^2-v^3+uv-u-v+1\\ &\quad=\ \tfrac{1}{4}\left[\left((u-v)^2+u+v-2\right)^2+3(u-v)^2(u+v-1)^2\right]\ \geq\ 0\,.\end{aligned}$$

Hence, $(u^3+v^3+1)^2+3(uv)^2-4(u^3v^3+u^3+v^3)\geq 0$.

Equality occurs if and only if $(u-v)^2+u+v-2=0$ and either $u=v$ or $u+v=1$. In the first case, we have $u=v$ and $u+v=2$; that is, $u=v=1$. In the second case, we have $u+v=1$ and $(u-v)^2=1$; that is, $(u,v)=(0,1)$ or $(u,v)=(1,0)$. In terms of x, y, and z, these conditions yield three solutions: (i) $x=y=z$, (ii) $x=0$, $y=z$, and (iii) $y=0$, $x=z$.

Therefore, in all cases,

$$(x^3+y^3+z^3)^2+3(xyz)^2\ \geq\ 4(y^3z^3+z^3x^3+x^3y^3)\,,$$

with equality if and only if one of the following occurs: (i) $x=y=z$, (ii) $x=0$ and $y=z$, (iii) $y=0$ and $x=z$, or (iv) $z=0$ and $x=y$.

II *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let

$$\begin{aligned}f(x,y,z)&=(x^3+y^3+z^3)^2+3(xyz)^2-4(y^3z^3+z^3x^3+x^3y^3)\\ &=x^6+(y^3-z^3)^2+3(xyz)^2-2z^3x^3-2x^3y^3\,.\end{aligned}$$

Clearly, $f(0,y,z)\geq 0$ with equality if and only if $y=z$. Symmetrically, $f(x,0,z)\geq 0$ and $f(x,y,0)\geq 0$ with equalities if and only if $z=x$ and $x=y$, respectively.

Assume henceforth that $xyz\neq 0$. Then $f(x,y,z)>0$ if both $zx<0$ and $xy<0$. Symmetrically, $f(x,y,z)>0$ for the other two cases where two of x, y, z have the same sign while the third has the opposite sign. Since $f(-x,-y,-z)=f(x,y,z)$ and since $f(x,y,z)$ is symmetric in its variables, we may now assume, without loss of generality, that $0<x\leq y\leq z$. Then

$$\begin{aligned}f(x,y,z)\ =\ &\tfrac{1}{16}\big[x^2(2x-y-z)^2(4x^2+3y^2+3z^2+4xy+6yz+4zx)\\ &+(y-z)^2(16y^4+32y^3z+48y^2z^2+32yz^3+16z^4\\ &-3x^2y^2-18x^2yz-3z^2x^2-24x^3y-24zx^3)\big]\ \geq\ 0\,.\end{aligned}$$

[*Ed*: This is true because $16y^4>3x^2y^2$, $32y^3z>24zx^3$, $48y^2z^2>18x^2yz$, $32yz^3>24x^3y$, and $16z^4>3z^2x^2$.]

Equality holds above if and only if $2x-y-z=0=y-z$; that is, if and only if either $x=y=z$ or two of x, y, z are equal while the third is 0.

Also solved by MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA (a second solution); and the proposer. There were seven incomplete *or partially incorrect solutions, most of which either made no mention of the cases for equality or claimed by mistake that "equality holds if and only if $x = y = z$." Some correspondents simply pointed out that the originally posed problem was incorrect.*

All the correct solutions with the exception of the two featured above and the ones by Hess and Janous, invoked Schur's Inequality or other known results. Zhou, in his second solution, remarked that the essential case when x, y, $z > 0$ has been proved in the book Mathematical Miniatures (MAA, 2003, Chap. 6, pp. 19–20) by S. Savchev and T. Andreescu, as an application of Popoviciu's Inequality: if f is a convex real-valued function defined on an interval I, then

$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \geq 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]$$

for all x, y, z in I.

2840. [2003 : 239] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let A' be an interior point of the line segment BC in $\triangle ABC$. The interior bisectors of $\angle BA'A$ and $\angle CA'A$ intersect AB and CA at D and E, respectively. Prove that AA', BE, and CD are concurrent.

Solution by almost everyone: Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; Jacques Choné, Nancy, France; Chip Curtis, Missouri Southern State College, Joplin, MO, USA; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; John G. Heuver, Grande Prairie, AB; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Geoffrey A. Kandall, Hamden, CT, USA; Murray S. Klamkin, University of Alberta, Edmonton, AB; Václav Konečný, Big Rapids, MI, USA; David Loeffler, student, Trinity College, Cambridge, UK; Gottfried Perz, Pestalozzigymnasium, Graz, Austria; Toshio Seimiya, Kawasaki, Japan; Bob Serkey, Leonia, NJ, USA; D.J. Smeenk, Zaltbommel, the Netherlands; Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany; Mihaï Stoënescu, Bischwiller, France; Peter Y. Woo, Biola University, La Mirada, CA, USA; Li Zhou, Polk Community College, Winter Haven, FL, USA; Titu Zvonaru, Bucharest, Romania; and the proposer.

An internal angle bisector of a triangle divides the opposite side in the ratio of the two other sides. Applying this to triangles $AA'B$ and $AA'C$ we get

$$\frac{AD}{DB} = \frac{|AA'|}{|BA'|} \quad \text{and} \quad \frac{CE}{EA} = \frac{|CA'|}{|AA'|}.$$

Hence,

$$\frac{AD}{DB} \cdot \frac{BA'}{A'C} \cdot \frac{CE}{EA} = \frac{AA'}{BA'} \cdot \frac{BA'}{A'C} \cdot \frac{A'C}{AA'} = 1.$$

From the converse to Ceva's Theorem, we see that AA', BB', and CC' are parallel or concurrent. As A', D, and E are interior points of the sides of triangle ABC, concurrency is the only possibility, since these three cevians must intersect one another in the interior of the triangle.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain, whose equally nice solution used Cartesian coordinates.

Loeffler reported that the locus of the intersection point as A' varies on BC is a smooth quartic curve passing through B and C and tangent to the angle bisectors of the given triangle at these points. [The graphics program Cinderella™ *suggests that if A is allowed to move along the entire line BC, then B and C are nodes of the quartic where it crosses itself at right angles.] In the case where $\triangle ABC$ is equilateral, the locus has equation $u^2(v^2+vw+w^2)=v^2w^2$ in trilinear coordinates u, v, w.*

2841. [2003 : 239] Corrected. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove the following inequalities:

$$\begin{aligned}
&\frac{\pi}{2}\left(1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}\right)\\
&\qquad\le\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}\\
&\qquad\le\frac{\pi}{2}\left(1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}+\frac{83}{2048n^4}\right).
\end{aligned}$$

Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina.

Since $(2n-1)!!=\dfrac{(2n)!}{(2n)!!}$ and $(2n)!!=2^n n!$, we can apply the well-known Stirling formula, $n!=n^n\sqrt{2\pi n}e^{-n+O(\frac{1}{n})}$, to obtain

$$\frac{(2n)!!}{(2n-1)!!}=\frac{(2^n n!)^2}{(2n)!}=\frac{2^{2n}n^{2n}(2\pi n)e^{-2n+O(\frac{1}{n})}}{(2n)^{2n}\sqrt{4\pi n}\,e^{-2n+O(\frac{1}{n})}}=\sqrt{\pi n}\,e^{O(\frac{1}{n})}.$$

Then

$$\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}=\frac{\pi n}{2n+1}e^{O(\frac{1}{n})};$$

whence,

$$\lim_{n\to\infty}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{2}{2n+1}=\pi.$$

Let

$$\begin{aligned}
f(n)&=\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{2}{2n+1},\\
g_1(n)&=1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}+\frac{83}{2048n^4},\\
\text{and}\quad g_2(n)&=1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}.
\end{aligned}$$

Since $\lim_{n\to\infty} g_1(n) = \lim_{n\to\infty} g_2(n) = 1$, we have

$$\lim_{n\to\infty} \frac{f(n)}{g_1(n)} = \lim_{n\to\infty} \frac{f(n)}{g_2(n)} = \pi\,. \tag{1}$$

Now, $\dfrac{f(n+1)}{f(n)} = \dfrac{4(n+1)^2}{(2n+1)(2n+3)} = \dfrac{4n^2+8n+4}{4n^2+8n+3}$, so that, using the inequalities

$$\begin{aligned}&(4n^2+8n+4)g_1(n) - (4n^2+8n+3)g_1(n+1)\\ &\quad= \frac{715n^4+1712n^3+2036n^2+1288n+332}{2048n^4(n+1)^4} > 0\end{aligned}$$

and

$$\begin{aligned}&(4n^2+8n+3)g_2(n+1) - (4n^2+8n+4)g_2(n)\\ &\quad= \frac{83n^3+168n^2+140n+44}{128n^3(n+1)^3} > 0\,,\end{aligned}$$

we obtain $\dfrac{f(n+1)}{f(n)} > \dfrac{g_1(n+1)}{g_1(n)}$ and $\dfrac{g_2(n+1)}{g_2(n)} > \dfrac{f(n+1)}{f(n)}$. Therefore, the sequence $\left\{\dfrac{f(n)}{g_1(n)}\right\}$ is strictly increasing, and the sequence $\left\{\dfrac{f(n)}{g_2(n)}\right\}$ is strictly decreasing. Since $g_1(n) > g_2(n)$, we have $\dfrac{f(n)}{g_1(n)} < \dfrac{f(n)}{g_2(n)}$. Finally, using the limits (1) and the fact that π is irrational, we obtain

$$\frac{\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{2}{2n+1}}{1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}+\frac{83}{2048n^4}} < \pi < \frac{\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{2}{2n+1}}{1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}}\,,$$

which completes the proof.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. All solvers have noticed that the original inequality is incorrect as stated and suggested the following correction: The original term $3/(32n^2)$ should have been $5/(32n^2)$ in both expressions in which it occurred. (This correction has been made in the problem statement above.) Then they proceeded with solving the corrected version.

2842. [2003 : 240]

Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

(a) $$\frac{\sum_{k=1}^{n} x_k^n}{n\prod_{k=1}^{n} x_k} + \frac{n\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \geq 2,$$

(b) $$\frac{\sum_{k=1}^{n} x_k^n}{\prod_{k=1}^{n} x_k} + \frac{\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \;\geq\; 1.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) By the Power-Mean Inequality, we have

$$\frac{1}{n}\sum_{k=1}^{n} x_k^n \;\geq\; \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)^n ,$$

and by the AM–GM Inequality, we get

$$t \;=\; \frac{\sum_{k=1}^{n} x_k}{\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}} \;\geq\; n\,.$$

Thus,

$$\frac{\sum_{k=1}^{n} x_k^n}{n\prod_{k=1}^{n} x_k} + \frac{n\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \;\geq\; \left(\frac{t}{n}\right)^n + \frac{n}{t} \;\geq\; 2\left(\frac{t}{n}\right)^{\frac{n-1}{2}} \;\geq\; 2\,.$$

Equality holds if and only if either $n = 1$ or $x_1 = x_2 = \cdots = x_n$ for $n \geq 2$.

(b) Similarly,

$$\begin{aligned}\frac{\sum_{k=1}^{n} x_k^n}{\prod_{k=1}^{n} x_k} + \frac{\left(\prod_{k=1}^{n} x_k\right)^{\frac{1}{n}}}{\sum_{k=1}^{n} x_k} \;&\geq\; n\left(\frac{t}{n}\right)^n + \frac{1}{t} \;\geq\; (n+1)\left(\frac{t}{n}\right)^{\frac{n^2}{n+1}}\left(\frac{1}{t}\right)^{\frac{1}{n+1}} \\ &=\; \frac{n+1}{n^{1/(n+1)}}\left(\frac{t}{n}\right)^{n-1} \;\geq\; \frac{n+1}{n^{1/(n+1)}}\,,\end{aligned}$$

where the second inequality makes use of the weighted AM–GM Inequality. Equality holds if and only if $n = 1$. Note also that

$$n\left(\frac{t}{n}\right)^n + \frac{1}{t} \;>\; n\,,$$

which is a better bound than $\frac{n+1}{n^{1/(n+1)}}$ when $n \geq 4$.

[*Ed.* It is easy to verify that $n\left(\frac{t}{n}\right)^n + \frac{1}{t} \geq n + \frac{1}{n}$, a lower bound which several solvers discovered.]

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

2843. [2003 : 240; corrected 2003 : 463] *Proposed by Bektemirov Baurjan, student, Aktobe, Kazakstan.*

Suppose that $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 4+\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}$ for positive real x, y, z. Prove that

$$(1-x)(1-y)(1-z) \;\leq\; \frac{1}{64}\,.$$

[*Editor's note*: The first printed version of this problem [2003 : 240] contained a misprint. Unfortunately, the problem intended by the proposer, which later appeared as a "correction" [2003 : 463], was still incorrect. Almost all the solvers who submitted solutions pointed this out by giving simple counterexamples, two of which are now given below.]

I *Solution by Michel Bataille, Rouen, France; Christopher J. Bradley, Bristol, UK; James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA; and Vedula N. Murty, Dover, PA, USA.*

As stated, the problem is wrong. Let $x=y=\frac{1}{4}$ and $z=\frac{3}{4}$. Then

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \;=\; 4+\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\,,$$

while $(1-x)(1-y)(1-z)=\frac{9}{64}>\frac{1}{64}$.

II *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Murray S. Klamkin, University of Alberta, Edmonton, AB.*

The problem is incorrect. Let $x=y=\varepsilon$, where $\varepsilon>0$ is arbitrarily small. Then the given condition is satisfied if we choose $z=4\varepsilon(1-\varepsilon)$. However, $(1-x)(1-y)(1-z)=(1-\varepsilon)^2(1-2\varepsilon)^2\to 1$ as $\varepsilon\to 0$.

Also solved by ∗ARKADY ALT, San Jose, CA, USA; ∗MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; NEVEN JURIČ, Zagreb, Croatia; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. (An asterisk ∗ indicates that the solver provided a counterexample to the first printed version only.) There were two incorrect solutions.

Alt remarked that if, in the corrected version, we add the condition that x, y, and z are the sides of a triangle, then the conclusion is true.

2844. [2003 : 241] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that the sequence $\{x_n\}$ satisfies $\sum\limits_{k=1}^{n} \frac{1}{k} - \ln(n + x_n) \;=\; \gamma$, where γ is Euler's constant.

(a) Prove that $\{x_n\}$ is convergent and that $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$.

(b) Determine an asymptotic approximation for the general term x_n, with an error that is $\mathrm{O}\left(\frac{1}{n^2}\right)$.

Solution by Roger Zarnowski, Angelo State University, San Angelo, TX, USA.

For each $n = 1, 2, \ldots$, let

$$\epsilon_n \;=\; \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma \,.$$

The given condition that defines the sequence $\{x_n\}$ may be rewritten as $\ln(1 + x_n/n) = \epsilon_n$. Solving for n, we get

$$x_n \;=\; n(e^{\epsilon_n} - 1)\,.$$

For part (a), we use the following inequalities from [1]:

$$\frac{1}{2(n+1)} \;<\; \epsilon_n \;<\; \frac{1}{2n}\,.$$

These imply that

$$n\left(e^{\frac{1}{2(n+1)}} - 1\right) \;<\; x_n \;<\; n\left(e^{\frac{1}{2n}} - 1\right)\,.$$

Expanding the exponentials, we obtain

$$n\left(\frac{1}{2(n+1)} + \mathrm{O}\left(\frac{1}{(n+1)^2}\right)\right) \;<\; x_n \;<\; n\left(\frac{1}{2n} + \mathrm{O}\left(\frac{1}{n^2}\right)\right)\,.$$

Thus, x_n is sandwiched between expressions which approach $\frac{1}{2}$ as $n \to \infty$. Therefore, the sequence $\{x_n\}$ is convergent, and $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$.

For part (b), we use the well-known asymptotic expansion

$$\epsilon_n \;=\; \frac{1}{2n} - \frac{1}{12n^2} + \mathrm{O}\left(\frac{1}{n^4}\right)\,.$$

See, for example, [2] (*Ed.* or [3] or [4]). We have

$$\begin{aligned} x_n \;=\; n(e^{\epsilon_n} - 1) \;&=\; n\left(\epsilon_n + \tfrac{1}{2}\epsilon_n^2 + \mathrm{O}\left(\epsilon_n^3\right)\right) \\ &=\; n\left(\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{2}\left(\frac{1}{2n}\right)^2 + \mathrm{O}\left(\frac{1}{n^3}\right)\right) \\ &=\; \frac{1}{2} + \frac{1}{24n} + \mathrm{O}\left(\frac{1}{n^2}\right)\,. \end{aligned}$$

[*Editor*: The result in part (a) is an immediate consequence of this asymptotic approximation for x_n. Therefore the proof of (a) given above is redundant. However, that proof has the advantage that it avoids using an asymptotic expansion for ϵ_n. The inequalities from [1] are, by comparsion, quite simple, and their proof in [1] is elementary.]

References

[1] R.M. Young, Euler's Constant, *Mathematical Gazette*, Vol. 75, No. 472 (June, 1991), pp. 187–190.

[2] Julian Havil, *Gamma*, Princeton Univ. Press, Princeton, NJ, 2003, p. 89.

[3] Ronald L. Graham, Donald E. Knuth and Oren Patashnik, *Concrete Mathematics*, Addison-Wesley, Boston, MA, formula (9.28), 1st ed. (1989), p. 438, or 2nd ed. (1994), p. 452.

[4] Vito Lampret, The Euler-Maclauin and Taylor Formulas: Twin, Elementary Derivations, *Mathematics Magazine*, vol. 74, No. 2 (April, 2001), pp. 109–122.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA (*part* (a) *only*)*; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; MIKE SPIVEY, Samford University, Birmingam, AL, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. One solution was spoiled by computational errors.*

Loeffler notes that $x_n = e^{\psi(n+1)} - n$*, where* $\psi(x)$ *is the digamma function, defined by* $\psi(x) = \frac{d}{dx}\Gamma(x)$*. He then bases his proof on the asymptotic expansion*

$$\psi(x) \;=\; \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^4}\right),$$

for which he gives the reference `http://functions.wolfram.com`.

Janous conjectures that the sequence $\{x_n\}$ *is strictly decreasing.* (*Can anyone supply a proof of this?*) *He heartily recommends the book* [*2*].

Reference [*4*] *was given by Bataille.*

2845. [2003 : 241] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let Q be a square of side length 1, and let S be a set consisting of a finite number of squares such that the sum of their areas is $\frac{1}{2}$.

Prove that the set S can be packed inside the square Q.

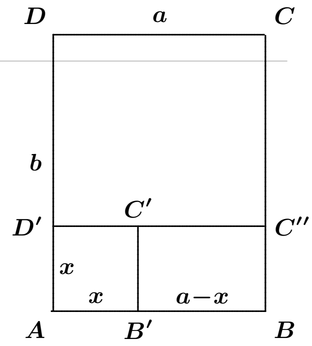
Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

This result can be generalized as follows:

Theorem. *Given a finite set S of square tiles and any rectangle R, then R can be packed with tiles from S up to at least half the area of R, so long as there are sufficient tiles in S small enough to fit into R.*

Proof. We shall use induction on the number n of tiles in S that can fit into R.

For $n = 1$, the theorem is true.



Assume that the theorem is true for $n = 1, 2, \ldots, k-1$. For case k, assume that S has k tiles that can fit into R. Let the vertices of R be A, B, C, D, and let $AB = a \leq b = AD$. Ignoring the tiles in S that will not fit, pick the largest tile and place it into R at vertex A. Call this $AB'C'D'$ where D' lies on AD, and let $AD' = x$. Now, consider the rectangle $R' = BC''C'B'$. It is large enough for any of the remaining $k-1$ tiles having size less than or equal to x. Thus, by the inductive assumption, it can be filled with tiles from S up to $\frac{1}{2}x(a-x)$ units of area. Again, because of the inductive assumption, the rectangle $D'DCC''$ can be tiled up to $\frac{1}{2}a(b-x)$ units of area, as long as there are sufficient remaining tiles. Therefore, as long as there are enough tiles of size less than or equal to a in S, the number of tiles used so far is x^2 plus half the remaining area of $ABCD$ not used up by $AB'C'D'$. This sum exceeds half of the area of $ABCD$.

Case k of the theorem is now proved, so that the theorem is true by induction.

The proposed problem follows as a corollary.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; and the proposer.

Guersenzvaig also proved the more general result.

2846. [2003 : 241] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

A regular simplex $S_n = A_1A_2A_3\ldots A_{n+1}$ is inscribed in the unit sphere Σ in $\mathbb{E}^n$. Let O be the origin in $\mathbb{E}^n$, $M \in \Sigma$, $u_k = \overrightarrow{OA_k}$ and $v = \overrightarrow{OM}$.

Find the maximum value of $\sum\limits_{k=1}^{n+1} |u_k \cdot v|$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

By symmetry, $\sum\limits_{k=1}^{n+1} u_k = 0$ and $u_i \cdot u_j = u_k \cdot u_l$ whenever $i \neq j$ and $k \neq l$. Thus,

$$0 = u_1 \cdot \sum_{k=1}^{n+1} u_k = ||u_1||^2 + \sum_{k=2}^{n+1}(u_1 \cdot u_k) = 1 + nu_1 \cdot u_2\,.$$

Hence, $u_i \cdot u_j = -\dfrac{1}{n}$ if $i \neq j$. Without loss of generality, we may assume that, for some $m \in \{1, \ldots, n\}$, we have

$$\begin{aligned} \sum_{k=1}^{n+1} |u_k \cdot v| &= \left(\sum_{k=1}^{m} u_k - \sum_{k=m+1}^{n+1} u_k \right) \cdot v = \left(2 \sum_{k=1}^{m} u_k \right) \cdot v \\ &\le 2 \left\| \sum_{k=1}^{m} u_k \right\| , \end{aligned}$$

with equality if and only if v is in the direction of $\sum\limits_{k=1}^{m} u_k$. Now

$$\begin{aligned} \left\| \sum_{k=1}^{m} u_k \right\|^2 &= \left(\sum_{k=1}^{m} u_k \right) \cdot \left(\sum_{k=1}^{m} u_k \right) = m + 2\binom{m}{2}\left(-\frac{1}{n}\right) \\ &= \frac{m(n-m+1)}{n} , \end{aligned}$$

which achieves a maximum at $m = \left\lfloor \frac{n+1}{2} \right\rfloor$. It follows that the desired maximum value of $\sum\limits_{k=1}^{n+1} |u_k \cdot v|$ is

$$2\sqrt{\frac{\lfloor \frac{n+1}{2} \rfloor \left(n - \lfloor \frac{n+1}{2} \rfloor + 1\right)}{n}} = \begin{cases} \dfrac{n+1}{\sqrt{n}} & \text{if } n \text{ is odd,} \\ \sqrt{n+2} & \text{if } n \text{ is even.} \end{cases}$$

Also solved by MICHEL BATAILLE, Rouen, France; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; and the proposer.

2847. [2003 : 242] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

The *inscircle* inscribed in a tetrahedron is a circle of maximum radius inscribed in the tetrahedron, considering every possible orientation in $\mathbb{E}^3$.

Find the radius of the inscircle of a regular tetrahedron.

Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The problem of determining the inscircle was solved around the end of the 19th century in an applied (technical) context, as far as I can remember. Maybe a reader can supply a reference. The desired radius equals

$$\frac{e}{2\sqrt{3}} ,$$

where e denotes the edge length of the regular tetrahedron. [Note that if Janous has correctly remembered the radius, it follows that the inscircle is inscribed in a face of the tetrahedron.]

Question for further research: What is the radius of the largest k–dimensional ball that can be inscribed in a regular simplex of n–dimensional space, $n \geq 4$ and $1 \leq k \leq n-1$?

2848. [2003 : 242] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB and K.R.S. Sastry, Bangalore, India.*

Suppose that A, B, and C are the angles of $\triangle ABC$ and that ω is its Crelle-Brocard angle. Prove that $A + \omega = \frac{\pi}{2}$ if and only if $\tan\ C$, $\tan\ A$, $\tan\ B$ are in geometric progression in that order.

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

The Crelle-Brocard angle satisfies $\cot\omega = \cot A + \cot B + \cot C$. [Some explanation is given at the end.] Since the given criterion $A + \omega = \pi/2$ is equivalent to $\cot\omega = \tan A$, we have

$$\tan A = \cot A + \cot B + \cot C\,. \qquad (1)$$

But we know that, for angles A, B, C satisfying $A + B + C = \pi$, we have

$$\frac{\tan A\ \tan B\ \tan C}{\tan A + \tan B + \tan C} = 1\,.$$

Let us therefore multiply the right side of equation (1) by this quantity. Our given equation $A + \omega = \pi/2$ holds if and only if

$$\tan A = \frac{\tan A\ \tan B + \tan B\ \tan C + \tan C\ \tan A}{\tan A + \tan B + \tan C}\,,$$

which happens if and only if

$$\tan A\ \tan B + \tan B\ \tan C + \tan C\ \tan A = \tan A(\tan A + \tan B + \tan C)\,,$$

and this occurs if and only if

$$\tan B\ \tan C = \tan^2 A\,.$$

This last equation is equivalent to $\tan C$, $\tan A$, $\tan B$ being in geometric progression, as required.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NEVEN JURIČ, Zagreb, Croatia; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposers.

The Crelle-Brocard points *(often called simply the* Brocard points*) are the pair of points denoted by* Ω *and* Ω' *that lie inside a triangle* ABC *and satisfy*

$$\angle\Omega AB = \angle\Omega BC = \angle\Omega CA, \quad \text{and} \quad \angle\Omega' BA = \angle\Omega' AC = \angle\Omega' CB.$$

One easily shows that these points are unique, that both triples of angles have the same measure ω*, and that* ω *satisfies* $\cot\omega = \cot A + \cot B + \cot C$*. A couple of solvers supplied proofs, four appealed to references (listed below), and the others accepted the properties as common knowledge. We thank Bataille, Janous, Jurić, and Zhou for supplying references.*

Our featured solution provides no hint at whether any triangles exist that satisfy $A + \omega = \pi/2$*. Jurić included an explicit example with his solution:*

$$a = 2, \quad b = \sqrt{2}, \quad c = \sqrt{2+\sqrt{8}}.$$

One computes here that $\tan A = \sqrt{1+\sqrt{8}}$, $\tan B = (\sqrt{2}-1)\sqrt{1+\sqrt{8}}$, *and* $\tan C = (\sqrt{2}+1)\sqrt{1+\sqrt{8}}$*. To answer the question of existence in general, substitute*

$$\tan C = -\tan(A+B) = \frac{\tan A + \tan B}{\tan A \, \tan B - 1}$$

(which is a simple consequence of $A+B+C=\pi$*) into our criterion* $\tan^2 A = \tan B \, \tan C$ *to get a quadratic equation for* $\tan B$ *in terms of* $\tan A$*. One finds a single solution when* $A = B = C = \pi/3$*, a pair of solutions when* $\pi/3 < A < \pi/2$*, and no solution for other values of* A*. In other words, if we take* $B \le C$*, then there is a unique triangle* ABC *that satisfies the given criterion when the measure of* A *lies in the interval* $\pi/3 \le A < \pi/2$*, and no triangle otherwise.*

References

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, page 266, Thm. 434.

[2] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989, pp. 59 (item (74)) and 360.

[3] Dominik Palman, article in *Matematika* (Zagreb), (July/September, 1984) p. 39.

[4] V.V. Prasolov, *Essays on Numbers and Figures*, AMS, 1997, Chapter 14.