A Maximum Vertical Angle II

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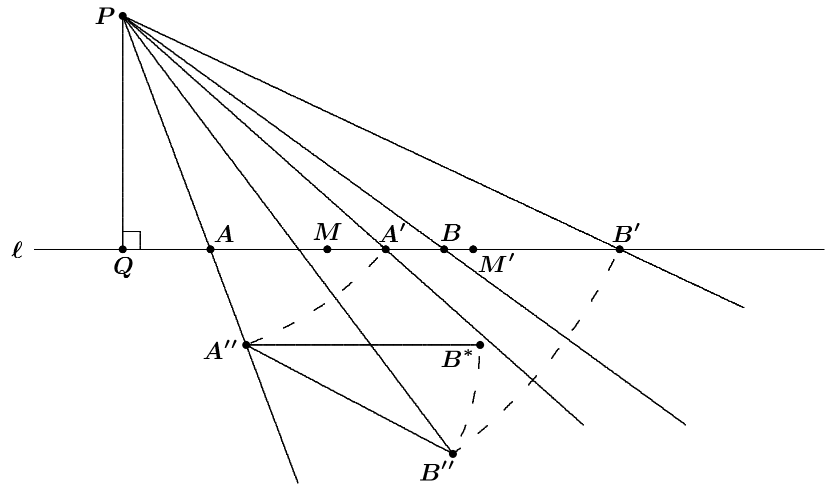
Theorem 1 On a line ℓ consider the congruent segments AB and $A'B'$ with mid-points M and M', respectively. Let P be a point not on ℓ, and let Q be the foot of the perpendicular from P to ℓ. Then $QM < QM'$ implies that $\angle A'PB' < \angle APB$.

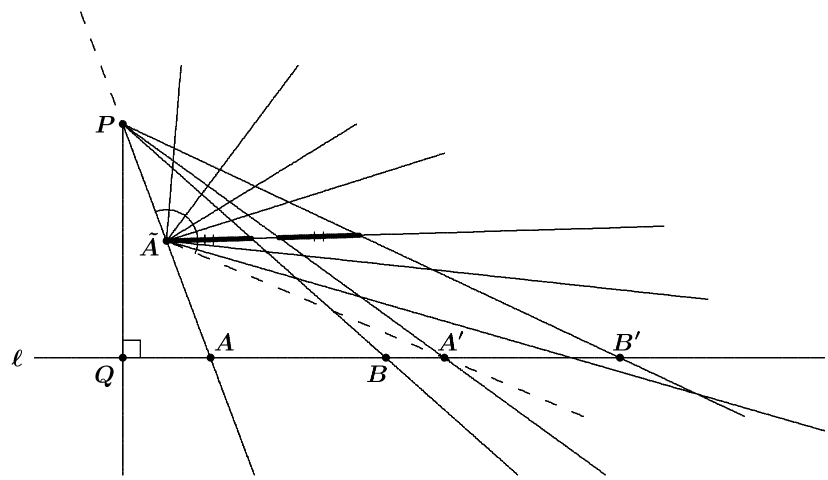
Proof. We first consider the case that AB and $A'B'$ belong to the same ray from Q with $QA < QB$ and $QA' < QB'$. We rotate $\triangle PA'B'$ about P into $\triangle PA''B''$, with A'' on the ray $\overrightarrow{PA}$.



Since $\angle A''PB'' = \angle A'PB'$, the proof is reduced to showing that B'' lies in the interior of $\angle APB$ (since that implies that $\angle APB > \angle A''PB''$). To this end, we translate segment AB along ray $\overrightarrow{PA}$ into $A''B^*$. The sides of an angle grow further apart as the distance from the vertex P increases, in both hyperbolic and Euclidean geometry. The same is true in spherical geometry for the first half of the distance from P to its antipodal point. For now, let us assume that $A''B^*$ does indeed lie sufficiently close to P; at the end we shall see that this assumption comes without loss of generality. Thus, in all the classical geometries, B^* lies in the interior of $\angle APB$. Note that with our assumption on Q, both $\angle PA''B^*$ $(= \angle PAB)$ and $\angle PA''B''$ $(= \angle PA'B')$ are at least $\pi/2$ with $\angle PA''B'' > \angle PA''B^*$. That, together with $A''B'' = A''B^*$, means that B'', just as B^*, belongs to the interior of $\angle APB$, as desired.

For the remaining cases we may assume without loss of generality that AB and $A'B'$ are disjoint—any overlap could be subtracted from both angles for the purpose of comparing them. Two situations have to be considered: that of Q between A and B, and that of Q between B and A'. (Note that we still assume $QA < QB$ and $QA' < QB'$.) Both cases are quickly reduced to the one that has already been handled. In the latter case, simply reflect $A'B'$ in the line PQ to the other side (so that AB and the image of $A'B'$ belong to the same ray from Q). In the former case, define B''' to be the point on $\overrightarrow{QB}$ that satisfies $QB''' = AB$, then prove that $\angle APB > \angle QPB''' > \angle A'PB'$.

Finally, we return to the question of the distance from the vertex P to the segment $A'B'$. Fix the sides of $\angle APB$ and $\angle A'PB'$ and choose a new position $\tilde{A}$ for A between A and P that is arbitrarily close to P. Consider the pencil of rays which pass through this new point $\tilde{A}$ and lie within $\angle A'\tilde{A}P$.



The portion of each line intercepted by the sides of $\angle A'PB'$ shrinks to zero from a segment longer than $A'B'$ $(= AB)$. Somewhere along the way, equal segments will be intercepted by both $\angle APB$ and $\angle A'PB'$. We began our proof with these two segments in place of the original pair.

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