A Maximum Vertical Angle I

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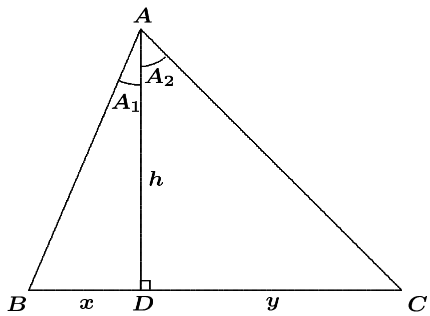
A problem proposed by C.N. Schmall [1] was to show that among all spherical triangles (convex) having the same base and equal altitudes, the isosceles triangle has the greatest vertical angle, and also to show that this was true for planar triangles. The solution by W.J. Thome was obtained via calculus. It turns out that, for the spherical case, there was the following prior non-calculus solution by W. Nicolls:



From the right triangle ADB, we get $\cot A_1 = \sin h \cot x$, and from right triangle ADC, we get $\cot A_2 = \sin h \cot y$. Thus,

$$\cot A \;=\; \cot\left(\cot^{-1}(\sin h \cot x) + \cot^{-1}(\sin h \cot y)\right) .$$

Since $\cot(u+v) = (\cot u \cot v - 1)/(\cot u + \cot v)$, we have

$$\cot A \;=\; \frac{\sin^2 h \cot x \cot y - 1}{\sin h(\cot x + \cot y)} \;=\; \frac{\sin^2 h \cos x \cos y - \sin x \sin y}{\sin h \sin a} ,$$

where we have set $a = x + y$, the length of the fixed base. Using the identities

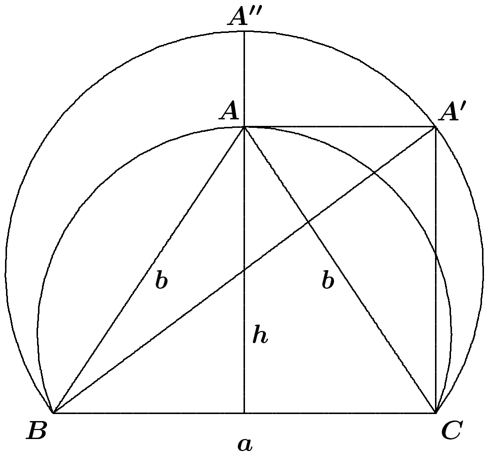
$$\begin{aligned} 2\cos x \cos y &= \cos(x-y) + \cos(x+y) , \\ 2\sin x \sin y &= \cos(x-y) - \cos(x+y) , \end{aligned}$$

we get

$$\cot A \;=\; \frac{(\cos a)(1+\sin^2 h) - \cos(x-y)\cos^2 h}{2 \sin h \sin a} .$$

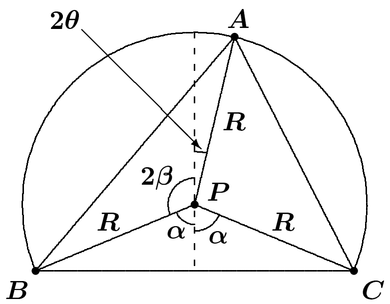
For A to be a maximum, we must have $\cot A$ a minimum; this occurs when $x = y$. Therefore, the triangle with the greatest vertical angle is isosceles.

We now give a solution that is guided by the following approach to the planar triangle case.



In the figure above, ABC is the isosceles triangle with given base a and altitude h. Let $A'BC$ be another triangle with the same base and altitude, and with A' on the same side of BC as A. Then the part of the circumcircle of $A'BC$ on the same side of BC as A' contains the part of the circumcircle of ABC on that side of BC. Since angles inscribed in a given segment are equal, $\angle A > \angle A'' = \angle A'$.

Applying this approach to the spherical case requires knowing how the angles inscribed in the same segment of a small circle behave. Consider now the above figure as the spherical case. First note that $\triangle A''BC$ is isosceles. The fact that $\angle A'' < \angle A$ follows from $\cot(A/2) = \sin h \cot(a/2)$ and $\cot(A''/2) = \sin h' \cot(a/2)$, where h' denotes the altitude of $A''BC$ and is less than $\pi/2$. Finally, we show that $\angle A' < \angle A''$.



$$\begin{aligned} \angle APB &= 2\beta + 2\theta \\ \angle APC &= 2\beta - 2\theta \\ \angle BPC &= 2\alpha \end{aligned}$$

In the figure above, ABC is a triangle inscribed in a segment of a small circle, where $\alpha + 2\beta = \pi$. We show that $\angle A$ is a decreasing function of θ. Since, in a right triangle with base angles x and α and hypotenuse R, one has $\cos R = \cot\alpha \cot x$, then, by dropping perpendiculars from P to the sides AB and AC, we see that

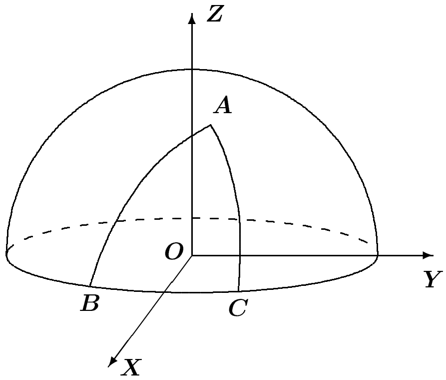
$$\cos R = \cot(\beta + \theta) \cot A_1 = \cot(\beta - \theta) \cot A_2 \,.$$

Hence,

$$\begin{aligned}\cot A &= \cot\left(\cot^{-1}\left(\frac{\cos R}{\cot(\beta+\theta)}\right)+\cot^{-1}\left(\frac{\cos R}{\cot(\beta-\theta)}\right)\right)\\ &= \frac{\cos^2 R-\cot(\beta+\theta)\cot(\beta-\theta)}{\cos R\big(\cot(\beta+\theta)+\cot(\beta-\theta)\big)}\\ &= \frac{(\cos^2 R)\sin(\beta+\theta)\sin(\beta-\theta)-\cos(\beta+\theta)\cos(\beta-\theta)}{\cos R\sin 2\beta}\\ &= \frac{-(\cos 2\beta)(1+\cos^2 R)-(\cos 2\theta)(\sin^2 R)}{2(\cos R)(\sin 2\beta)}\end{aligned}$$

As θ increases from 0, the value of $-\cos 2\theta$ increases, implying that A decreases.

To better understand what is going on here, we use coordinates. We position the spherical triangle on the sphere $x^2+y^2+z^2=1$, as shown below, in such a way that $C=(a,b,0)$, $B=(a,-b,0)$, and $A=(x,y,z)$, where a, $b>0$ and $a^2+b^2=1$. Using standard spherical coordinates, we have $x=\cos\theta\sin\phi$, $y=\sin\theta\sin\phi$, and $z=\cos\phi$. If the angle between OB and OC is 2α, then $a=\cos\alpha$ and $b=\sin\alpha$.



The formula used in the above solution by Nicolls takes the form

$$\cot A = \frac{\cos^2\phi\cos(\alpha+\theta)\cos(\alpha-\theta)-\sin(\alpha+\theta)\sin(\alpha-\theta)}{\cos\phi\sin 2\alpha}.$$

This can be simplified to

$$\cot A = \frac{y^2+(a^2z^2-b^2)}{2abz}. \tag{1}$$

The latter formula can also be established directly with some vector algebra. It is clear from (1) that, with $z>0$ kept constant (so that spherical triangle ABC has constant altitude), $\cot A$ is minimized when $y=0$; whence, $\angle A$ is maximized when ABC is isosceles.

The formula (1) also gives some insight into the behavior of $\angle A$ as the point $A(x, y, z)$ varies over the sphere with B and C fixed. For instance, the locus of points A such that $\angle A$ is constant satisfies

$$y^2 + a^2 z^2 - b^2 \;=\; 2abkz \,,$$

where $x^2 + y^2 + z^2 = 1$ and $k = \cot A$ is a constant. The projection of this spherical curve onto the yz–plane is an arc of an ellipse centered at $(0, bk/a)$ and having semi-minor axis $b\sqrt{1+k^2}$ and semi-major axis $b\sqrt{1+k^2}/a$. This ellipse passes through $(b, 0)$ and $(-b, 0)$.

The locus of points A for which $\angle A = 90°$ is of interest. In this case, we have $k = \cot A = 0$, and the projection onto the yz–plane has the equation $y^2 + a^2 z^2 = b^2$, an ellipse centered at $(0, 0)$ with semi-minor axis b and semi-major axis b/a. In particular, if $a = b = 1/\sqrt{2}$, the projection is the ellipse $2y^2 + z^2 = 2$. Note that in this case the locus itself on the sphere consists of two perpendicular arcs of great circles joining B and C to the north pole $(0, 0, 1)$!

As an exercise, the reader may verify that for $\angle A = 90°$, the projection of the locus onto the xy–plane is an arc of the hyperbola having equation

$$a^2 x^2 - b^2 y^2 \;=\; a^2 - b^2 \,.$$

When $a = b = 1/\sqrt{2}$, this degenerates into a pair of perpendicular straight lines.

Remark: The referee has noted some previously unpublished work of Dieter Ruoff and J. Chris Fisher in which our result is proved as one in absolute geometry. Ruoff and Fisher's proof follows immediately as part II.

Reference:

[1] C.N. Schmall, Problem 414, Amer. Math. Monthly **24:3** (1917), 185–186.

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