THE OLYMPIAD CORNER

No. 238

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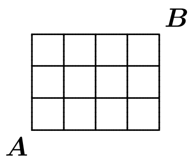
The first problem set we give this issue is the National Round of the XXXVI Spanish Mathematical Olympiad of the Real Sociedad Matemática Española. Thanks go to Christopher Small, Canadian Team Leader to the 42nd IMO, for collecting these problems for our use.

XXXVI SPANISH MATHEMATICAL OLYMPIAD
National Round
First Day

1. Let $P(x) = x^4 + ax^3 + bx^2 + cx + 1$ and $Q(x) = x^4 + cx^3 + bx^2 + ax + 1$, with a, b, c real numbers and $a \neq c$. Find conditions on a, b, and c so that $P(x)$ and $Q(x)$ have two common roots. In this case, solve the equations $P(x) = 0$, $Q(x) = 0$.

2. The figure shows a street plan of twelve square blocks. A person P goes from point A to point B, and a second person Q goes from B to A. Both of them (P and Q) leave at the same time with the same speed, following shortest paths on the grid. At each corner they choose among the possible streets with equal probability. What is the probability that P meets Q?



3. Circles C_1 and C_2 intersect at points A and B. A line r through B intersects C_1 and C_2 again at points P_r and Q_r, respectively. Prove that there is a point M, which depends only on C_1 and C_2, such that the perpendicular bisector of P_rQ_r passes through M.

Second Day

4. For any integer x, let $\lfloor x \rfloor$ denote the integer part of x. Find the largest integer N satisfying the following conditions:

(a) $\left\lfloor \frac{N}{3} \right\rfloor$ has three identical digits, and

(b) $\left\lfloor \frac{N}{3} \right\rfloor$ is the sum of n consecutive positive integers starting at 1; that is, there is a positive integer n such that

$$\left\lfloor \frac{N}{3} \right\rfloor = 1 + 2 + \cdots + (n-1) + n\,.$$

5. Four points are placed in a square of side 1. Show that the distance between some two of them is less than or equal to 1.

6. Show that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n + 1$.

As a second set this issue we give the 2000 Taiwan Mathematical Olympiad. Thanks again go to Christopher Small.

TAIWAN (ROC) MATHEMATICAL OLYMPIAD
Part I – April 7, 2000
Time: 4.5 hours

1. Find all pairs (x, y) of positive integers such that $y^{x^2} = x^{y+2}$.

2. In an acute triangle ABC with $|AC| > |BC|$, let M be the mid-point of AB. Let AP be the altitude from A and BQ be the altitude from B. These altitudes meet at H, and the lines AB and PQ meet at R. Prove that the two lines RH and CM are perpendicular.

3. Let $S = \{1, 2, 3, \ldots, 100\}$, and let P denote the family of all subsets T of S such that $|T| = 49$. We assign to each set T in P a label from the set $\{1, 2, \ldots, 100\}$, chosen at random. Show that there exists a subset M of S, with $|M| = 50$, such that for each $x \in M$, the label of $M\backslash\{x\}$ is not equal to x.

Part 2 — April 29, 2000
Time: 4.5 hours

4. For each positive integer k, let $\varphi(k)$ denote the number of positive integers n satisfying $\gcd(n, k) = 1$ and $n \le k$. Suppose that $\varphi(5^m - 1) = 5^n - 1$ for some positive integers m, n. Prove that $\gcd(m, n) > 1$.

5. Let $A = \{1, 2, 3, \ldots, n\}$, where n is a positive integer. A subset of A is *connected* if it consists of one element or some consecutive integers. Determine the greatest integer k for which A contains k distinct subsets A_1, A_2, ..., A_k such that the intersection of any two sets A_i and A_j is connected.

6. Let $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be defined by

$$f(1) = 0, \quad \text{and} \quad f(n) = \max_{1 \le j \le \lfloor \frac{n}{2} \rfloor} \{f(j) + f(n-j) + j\}, \quad \forall n \ge 2.$$

Determine $f(2000)$.

As a third set of problems, we give the first and final rounds of the 2000 Hungarian National Olympiad for specialized math classes. Thanks go to Christopher Small, Canadian Team Leader to the 42$^{\text{nd}}$ IMO, for collecting them for our use.

2000 HUNGARIAN NATIONAL OLYMPIAD
Specialized Math Classes

First Round

1. Let x, y, and z denote positive real numbers, each less than 4. Prove that at least one of the numbers $\frac{1}{x}+\frac{1}{4-y}$, $\frac{1}{y}+\frac{1}{4-z}$, and $\frac{1}{z}+\frac{1}{4-x}$ is greater than or equal to 1.

2. Find the integer solutions of $5x^2 - 14y^2 = 11z^2$.

3. Find the triangles for which the median and altitude starting from the same vertex are symmetrical to the angle bisector starting from the same vertex.

4. If $1 \le m \le n$, prove that m is a divisor of

$$n\left(\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^{m-1}\binom{n}{m-1}\right).$$

5. Find the smallest real number c with the following property: On the perimeter of any triangle, there are two points, separated by a distance of at most c times the perimeter, that divide the perimeter into two equal parts.

Final Round

1. Let c denote a positive integer, and let c_1, c_3, c_7, and c_9 be the number of divisors of c which have last digit 1, 3, 7, and 9, respectively (in the decimal system). Prove that $c_3 + c_7 \le c_1 + c_9$.

2. Circles k_1 and k_2 and a point P are given in a plane. Construct a line passing through P which meets the circle k_i at A_i and B_i in such a way that there exist points C_i on k_i such that $A_1C_1 = B_1C_1 = A_2C_2 = B_2C_2$. (It is not necessary to find the number of solutions, nor the condition for the existence of such a line.)

3. We have integers greater than 1, denoted by $a_1, \ldots, a_k, b_1, \ldots, b_m$. Every a_i is the product of an even number of (not necessarily distinct) primes. We have chosen some integers from the $k+m$ given integers (possibly none or all of them) such that every b_i has an even number of divisors among the chosen integers. In how many ways can we make such a choice?

As a final set of problems for your puzzling pleasure over the (Canadian) summer, we give the problems of the 2000 Kürschák Contest from Hungary. Thanks again go to Christopher Small for providing these problems.

2000 KÜRSCHÁK CONTEST

1. For a positive integer n, consider the square in the Cartesian plane whose vertices are $A(0,0)$, $B(n,0)$, $C(n,n)$ and $D(0,n)$. The grid points of the integer lattice inside or on the boundary of this square are coloured either red or green in such a way that every unit square in the lattice has exactly two red vertices. How many such colourings are possible?

2. Let T be a point in the plane of the non-equilateral triangle ABC which is different from the vertices of the triangle. Let the lines AT, BT, and CT meet the circumcircle of the triangle at A_T, B_T, and C_T, respectively. Prove that there are exactly two points P and Q in the plane for which the triangles $A_PB_PC_P$ and $A_QB_QC_Q$ are equilateral. Prove, furthermore, that the line PQ passes through the circumcentre of the triangle ABC.

3. Let k denote a non-negative integer. Assume that the integers $a_1, \ldots, a_n$ give at least $2k$ different remainders when divided by $n+k$. Prove that some of the integers add up to a number divisible by $n+k$.

An error has been pointed out by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. In the statement of Problem #6 of the 2001 Hungary-Israel Binational Mathematics Competition [2004 : 82], the word "one" should be "none", since the disjoint subsets in the problem do not exist if any of the numbers is greater than 60.

We now turn our attention to readers' solutions to problems of the February 2002 number of the *Corner*. First we have solutions to problems of the Vietnamese Mathematical Olympiad 1999, Category A [2002 : 5–6].

1. Solve the system of equations

$$\begin{cases} (1+4^{2x-y})5^{1-2x+y} &= 1+2^{2x-y+1} \\ y^3+4x+1+\ln(y^2+2x) &= 0 \end{cases}$$

Solved by Mohammed Aassila, Strasbourg, France; and Pavlos Maragoudakis, Lefkogia, Crete, Greece. We give the write-up of Maragoudakis.

Letting $t = 2x - y$, the first equation becomes $f(t) = 0$, where

$$f(t) \;=\; 1 + 2\cdot 2^t - 5\cdot\left(\tfrac{1}{5}\right)^t - 5\cdot\left(\tfrac{4}{5}\right)^t\,.$$

Since the function f is strictly increasing, the obvious solution $t = 1$ is the only solution. Thus, $2x - y = 1$; that is, $y = 2x - 1$.

The second equation then becomes $g(x) = 0$, where

$$g(x) \;=\; (2x-1)^3 + 4x + 1 + \ln(4x^2 - 2x + 1)\,.$$

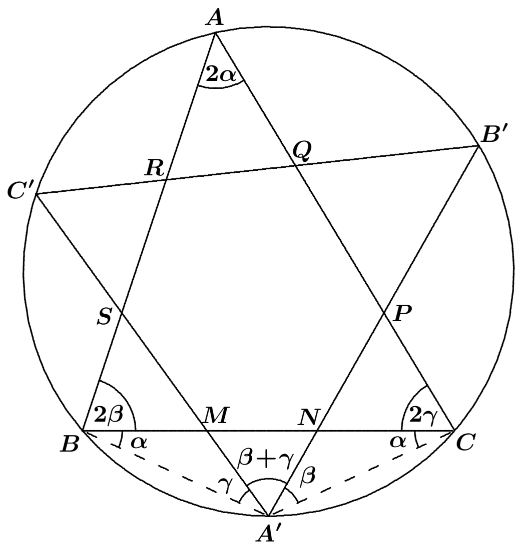
We easily find

$$g'(x) \;=\; 6(2x-1)^2 + \frac{16x^2+2}{3x^2+(x-1)^2} \;>\; 0\,.$$

Therefore, g is strictly increasing, and the obvious solution $x = 0$ is the only solution. Finally, $(x, y) = (0, -1)$.

2. Let A', B', C' be the respective mid-points of the arcs BC, CA, AB, not containing points A, B, C, respectively, of the circumcircle of the triangle ABC. The sides BC, CA, and AB intersect the pairs of segments $(C'A', A'B')$, $(A'B', B'C')$, and $(B'C', C'A')$ at the pairs of points (M, N), (P, Q), and (R, S), respectively. Prove that $MN = PQ = RS$ if and only if the triangle ABC is equilateral.

Solved by Christopher J. Bradley, Bristol, UK; Geoffrey A. Kandall, Hamden, CT, USA; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's write-up.



We put $\angle BAC = 2\alpha$, $\angle ABC = 2\beta$, and $\angle ACB = 2\gamma$. Then $\alpha + \beta + \gamma = \frac{\pi}{2}$. Since A' is the mid-point of the arc BC, we have

$$\angle A'AB \;=\; \angle A'CB \;=\; \angle A'BC \;=\; \angle A'AC \;=\; \alpha\,.$$

Also, $\angle B'A'C = \angle B'BC = \beta$ and $\angle BA'C' = \angle BCC' = \gamma$. Hence,

$$\angle A'NM \;=\; \angle A'CB + \angle B'A'C \;=\; \alpha + \beta\,,$$

and

$$\angle A'MN = \angle A'BC + \angle BA'C' = \alpha + \gamma .$$

Then

$$\begin{aligned}\angle MA'N &= 180^\circ - \angle A'NM - \angle A'MN \\ &= 180^\circ - (2\alpha + \beta + \gamma) = \beta + \gamma .\end{aligned}$$

Let R be the circumradius of $\triangle ABC$. Then

$$BA' = 2R \sin \angle A'CB = 2R \sin \alpha . \tag{1}$$

Applying the Law of Sines to $\triangle A'BM$ and $\triangle A'MN$, we get

$$\frac{MA'}{BA'} = \frac{\sin \angle A'BM}{\sin \angle A'MB} = \frac{\sin \alpha}{\sin(\pi - \alpha - \gamma)} = \frac{\sin \alpha}{\sin(\alpha + \gamma)} = \frac{\sin \alpha}{\cos \beta} ,$$

and

$$\frac{MN}{MA'} = \frac{\sin \angle MA'N}{\sin \angle A'NM} = \frac{\sin(\beta + \gamma)}{\sin(\alpha + \beta)} = \frac{\cos \alpha}{\cos \gamma} .$$

Hence,

$$\frac{MN}{BA'} = \frac{MN}{MA'} \cdot \frac{MA'}{BA'} = \frac{\cos \alpha}{\cos \gamma} \cdot \frac{\sin \alpha}{\cos \beta} . \tag{2}$$

It follows from (1) and (2) that

$$MN = \frac{\sin \alpha \cos \alpha}{\cos \beta \cos \gamma} \cdot BA' = 2R \frac{\sin^2 \alpha \cos \alpha}{\cos \beta \cos \gamma} = \frac{R(\sin 2\alpha)^2}{2 \cos \alpha \cos \beta \cos \gamma} .$$

Similarly, we have

$$PQ = \frac{R(\sin 2\beta)^2}{2 \cos \alpha \cos \beta \cos \gamma} \quad \text{and} \quad RS = \frac{R(\sin 2\gamma)^2}{2 \cos \alpha \cos \beta \cos \gamma} .$$

Thus, $MN = PQ = RS$ if and only if $(\sin 2\alpha)^2 = (\sin 2\beta)^2 = (\sin 2\gamma)^2$, which is true if and only if $\sin 2\alpha = \sin 2\beta = \sin 2\gamma$. This is equivalent to $\alpha = \beta = \gamma$, which means that $\triangle ABC$ is equilateral.

3. Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be two sequences defined recursively as follows:

$$\begin{aligned} &x_0 = 1 , \quad x_1 = 4 , \quad x_{n+2} = 3x_{n+1} - x_n , \\ &y_0 = 1 , \quad y_1 = 2 , \quad y_{n+2} = 3y_{n+1} - y_n , \end{aligned}$$

for all $n = 0, 1, 2, \ldots$.

(a) Prove that

$$x_n^2 - 5y_n^2 + 4 = 0$$

for all non-negative integers n.

(b) Suppose that a, b are two positive integers such that $a^2 - 5b^2 + 4 = 0$. Prove that there exists a non-negative integer k such that $x_k = a$ and $y_k = b$.

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We first give Bataille's treatment of the problem.

(a) The sequences $\{x_n\}$, $\{y_n\}$ may be defined equivalently by

$$x_0 = y_0 = 1, \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \tag{1}$$

where A denotes the matrix $\begin{pmatrix} 3/2 & 5/2 \\ 1/2 & 3/2 \end{pmatrix}$. Indeed, (1) gives $x_1 = 4$, $y_1 = 2$,

$$\begin{aligned} x_{n+2} &= \tfrac{3}{2}x_{n+1} + \tfrac{5}{2}y_{n+1} = \tfrac{3}{2}x_{n+1} + \tfrac{5}{2}\left(\tfrac{1}{2}x_n + \tfrac{3}{2}y_n\right) \\ &= \tfrac{3}{2}x_{n+1} + \tfrac{5}{4}x_n + \tfrac{3}{2}\left(x_{n+1} - \tfrac{3}{2}x_n\right) = 3x_{n+1} - x_n, \end{aligned}$$

and $y_{n+2} = 3y_{n+1} - y_n$ (similarly). Now, the required result follows by induction, using the relations $x_0^2 - 5y_0^2 + 4 = 0$ and

$$\begin{aligned} x_{n+1}^2 - 5y_{n+1}^2 + 4 &= \left(\tfrac{3}{2}x_n + \tfrac{5}{2}y_n\right)^2 - 5\left(\tfrac{1}{2}x_n + \tfrac{3}{2}y_n\right)^2 + 4 \\ &= x_n^2 - 5y_n^2 + 4. \end{aligned}$$

(b) Let a and b be positive integers such that $a^2 - 5b^2 + 4 = 0$. Clearly, $a = b = 1$ if $a = 1$ or $b = 1$. Also, we see that a and b have the same parity. Now, suppose $a > 1$ and $b > 1$. Let $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = A^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$; that is,

$$a_1 = \frac{3a - 5b}{2} \quad \text{and} \quad b_1 = \frac{-a + 3b}{2}.$$

Then a_1 and b_1 are integers, and $a_1^2 - 5b_1^2 + 4 = a^2 - 5b^2 + 4 = 0$. We have $a - a_1 = \dfrac{5b - a}{2}$ and $b - b_1 = \dfrac{a - b}{2}$. The following calculations show that a_1, b_1, $a - a_1$, and $b - b_1$ are all positive:

$$\begin{aligned} (3a - 5b)(a + 3b) &= 4(ab - 3) > 0, \\ (3b - a)(3a + 5b) &= 4(ab + 3) > 0, \\ (5b - a)(5b + a) &= 25b^2 - a^2 = 4(a^2 + 5) > 0, \\ (a - b)(a + b) &= a^2 - b^2 = 4(b^2 - 1) > 0. \end{aligned}$$

If $a_1 = 1$ or $b_1 = 1$, then $a_1 = b_1 = 1 = x_0 = y_0$ and

$$\begin{pmatrix} a \\ b \end{pmatrix} = A\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

Otherwise, we iterate the process until we get $a_k = 1 = b_k$ (which will necessarily occur, since we have decreasing sequences of positive integers). Then

$$\begin{pmatrix} a \\ b \end{pmatrix} = A^k\begin{pmatrix} a_k \\ b_k \end{pmatrix} = A^k\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix};$$

that is, $a = x_k$ and $b = y_k$.

Next we give Bornsztein's write-up, which points out some interesting connections.

(a) Solving the recurrence relations, we find that, for all $n = 1, 2, \ldots,$

$$\begin{aligned} x_n &= \varphi^{2n-1} + \left(-\frac{1}{\varphi}\right)^{2n-1} \\ y_n &= \frac{1}{\sqrt{5}}\varphi^{2n-1} - \frac{1}{\sqrt{5}}\left(-\frac{1}{\varphi}\right)^{2n-1}, \end{aligned}$$

where $\varphi = \dfrac{1+\sqrt{5}}{2}$ is the Golden Ratio. Using Binet's Formula, we deduce that

$$x_n = L_{2n-1} \qquad \text{and} \qquad y_n = f_{2n-1}\,,$$

where $\{f_n\}_{n=1}^{\infty}$ is the Fibonacci sequence (defined by $f_1 = 1, f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$, for all $n = 1, 2, \ldots$), and $\{L_n\}_{n=1}^{\infty}$ is the Lucas sequence (defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$, for all $n = 1, 2, \ldots$).

Using Binet's Formula, it is easy to verify that, for all $n = 1, 2, \ldots,$

$$5f_n^2 + 4(-1)^n \;=\; L_n^2\,. \tag{1}$$

It follows that $x_n^2 - 5y_n^2 + 4 = 0$.

(b) The following theorem is known (see [1]):

Theorem. Either $5x^2 + 4 = y^2$ or $5x^2 - 4 = y^2$ has a solution (x, y) in positive integers if and only if $(x, y) = (f_n, L_n)$ for some n..

From (1), we have $5f_{2n}^2 + 4 = L_{2n}^2$. Consequently, the solutions of $5x^2 - 4 = y^2$ in positive integers are the pairs $(x, y) = (f_{2n-1}, L_{2n-1})$, where $n = 1, 2, \ldots$. Then the solutions of $a^2 - 5b^2 + 4 = 0$ in positive integers are the pairs $(a, b) = (x_n, y_n)$, where $n = 1, 2, \ldots$.

Reference

[1] R. Honsberger, *Mathematical Gems III*, MAA, p. 115.

4. Let a, b, c be real positive numbers such that $abc + a + c = b$. Determine the greatest possible value of the following expression

$$P \;=\; \frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1}\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bataille's solution.

Let a, b, c be real positive numbers such that $abc + a + c = b$. Let $\alpha = \tan^{-1} a$ and $\beta = \tan^{-1} b$. Then $\alpha, \beta \in (0, \frac{\pi}{2})$ and $\alpha < \beta$ (since $a < b$). Thus, we have $a = \tan\alpha$, $b = \tan\beta$, and $c = \dfrac{b-a}{1+ab} = \tan(\beta - \alpha)$.

We may rewrite P as

$$\begin{aligned} P &= 2\cos^2\alpha - 2\cos^2\beta + 3\cos^2(\beta-\alpha) \\ &= 3\cos^2(\beta-\alpha) + 2\sin(\beta+\alpha)\sin(\beta-\alpha)\,. \\ &\le 3\cos^2(\beta-\alpha) + 2\sin(\beta-\alpha) \;=\; f(\beta-\alpha)\,, \end{aligned}$$

where $f(x) = 3\cos^2 x + 2\sin x$. Since $f'(x) = 2\cos x(1 - 3\sin x)$, the maximum value of f on the interval $(0, \frac{\pi}{2})$ is attained at $x_0 = \sin^{-1}(\frac{1}{3})$. The maximum is $f(x_0) = 3(1-\frac{1}{9}) + \frac{2}{3} = \frac{10}{3}$. It follows that $P \le \frac{10}{3}$, the value $\frac{10}{3}$ being attained when $\beta+\alpha = \frac{\pi}{2}$ and $\beta - \alpha = x_0$.

These conditions on α and β yield $2\alpha = \frac{\pi}{2} - x_0$ and $2\beta = \frac{\pi}{2} + x_0$. Hence,

$$\frac{1-\tan^2\alpha}{1+\tan^2\alpha} \;=\; \cos 2\alpha \;=\; \cos\left(\frac{\pi}{2} - x_0\right) \;=\; \sin x_0 \;=\; \frac{1}{3}\,,$$

from which it is easy deduce that $\tan\alpha = \dfrac{\sqrt{2}}{2}$; that is, $a = \dfrac{\sqrt{2}}{2}$. Then $b = \tan\left(\dfrac{\pi}{2} - \alpha\right) = \cot\alpha = \sqrt{2}$, and $c = \tan(\beta-\alpha) = \dfrac{\sqrt{2}}{4}$.

In conclusion, the maximum value of P, under the specified conditions, is $\dfrac{10}{3}$, which is attained when $a = \dfrac{\sqrt{2}}{2}$, $b = \sqrt{2}$, and $c = \dfrac{\sqrt{2}}{4}$.

5. In three-dimensional space, let Ox, Oy, Oz, Ot be four non-planar distinct rays such that the angles between any two of them have the same measure.

(a) Determine this common measure.

(b) Let Or be another ray different from the above four rays. Let α, β, γ, δ be the angles formed by Or with Ox, Oy, Oz, Ot, respectively. Put

$$p \;=\; \cos\alpha + \cos\beta + \cos\gamma + \cos\delta\,,$$

$$q \;=\; \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta\,.$$

Prove that p and q are invariant when Or rotates about the point O.

Solved by Christopher J. Bradley, Bristol, UK; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Klamkin's solution and generalization.

(a) This is a known result which has even appeared in this journal for $n+1$ rays in E^n.

Let $XYZT$ be a regular tetrahedron with centre O and circumradius 1. Let the four unit vectors from O to X, Y, Z, and T be denoted by x, y, z, and t, respectively. These four vectors make equal angles with each other. Also, $\mathrm{x}+\mathrm{y}+\mathrm{z}+\mathrm{t} = 0$, since O is the centroid. Expanding $|\mathrm{x}+\mathrm{y}+\mathrm{z}+\mathrm{t}|^2 = 0$, we get $4 + 12\cos\theta = 0$, or $\theta = \cos^{-1}(-1/3)$, where θ is the angle between any two of the vectors.

(b) Let v be the vector of the form $c_0\mathrm{x} + c_1\mathrm{y} + c_2\mathrm{z} + c_3\mathrm{t}$ such that v is parallel to the ray Or and $\sum_{k=0}^{3} c_k = 1$. It will suffice to show that p and q are independent of c_0, c_1, c_2, and c_3. We have

$$\cos\alpha \;=\; \mathrm{x}\cdot\frac{\mathrm{v}}{|\mathrm{v}|} \;=\; \frac{c_0 - \frac{1}{3}(c_1+c_2+c_3)}{|\mathrm{v}|} \;=\; \frac{4c_0 - 1}{3|\mathrm{v}|}\,,$$

with similar expressions for $\cos\beta$, $\cos\gamma$, and $\cos\delta$. Also,

$$\begin{aligned} |\mathrm{v}|^2 &= \sum_{k=0}^{3} c_k^2 - \frac{2}{3}\sum_{i<j} c_i c_j \\ &= \sum_{k=0}^{3} c_k^2 - \frac{1}{3}\left(1 - \sum_{k=0}^{3} c_k^2\right) \;=\; \frac{1}{3}\left(4\sum_{k=0}^{3} c_k^2 - 1\right). \end{aligned}$$

Hence, $p = \frac{1}{3|\mathrm{v}|}\sum_{k=0}^{3}(4c_k - 1) = 0$, and

$$\begin{aligned} q &= \frac{1}{9|\mathrm{v}|^2}\sum_{k=0}^{3}(4c_k-1)^2 \;=\; \frac{1}{9|\mathrm{v}|^2}\left(16\sum_{k=0}^{3} c_k^2 - 8\sum_{k=0}^{3} c_k + 4\right) \\ &= \frac{4}{9|\mathrm{v}|^2}\left(4\sum_{k=0}^{3} c_k^2 - 1\right) \;=\; \frac{4}{3}\,. \end{aligned}$$

These results generalize to E^n, where we have $n+1$ concurrent rays such that the angle between any two of them has the same measure. We start with $n+1$ unit vectors, $\mathrm{x_0}$, $\mathrm{x_1}$, …, $\mathrm{x_n}$, from the centroid to the vertices of a regular simplex $X_0X_1\cdots X_n$. Expanding $|\mathrm{x_0} + \mathrm{x_1} + \cdots + \mathrm{x_n}|^2 = 0$ gives

$$n + 1 + 2\binom{n+1}{2}\cos\theta \;=\; 0\,,$$

which yields $\cos\theta = -1/n$, or $\theta = \cos^{-1}(-1/n)$.

Next, let $\mathrm{v} = c_0\mathrm{x_0} + c_1\mathrm{x_1} + \cdots + c_n\mathrm{x_n}$ such that the angle between v and $\mathrm{x_k}$ is α_k and $\sum_{k=0}^{n} c_k = 1$. Then

$$\cos\alpha_k \;=\; \mathrm{x_k}\cdot\frac{\mathrm{v}}{|\mathrm{v}|} \;=\; \frac{(n+1)c_k - 1}{n|\mathrm{v}|} \quad\text{and}\quad n|\mathrm{v}|^2 \;=\; (\mathrm{n}+1)\sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{c_k^2} - 1\,.$$

Hence, $\sum_{k=0}^{n}\cos\alpha_k = 0$ and

$$\sum_{k=0}^{n}\cos^2\alpha_k \;=\; \frac{(n+1)\left((n+1)\sum_{k=0}^{n} c_k^2 - 1\right)}{n^2|\mathrm{v}|^2} \;=\; \frac{n+1}{n}\,.$$

Next we turn to problems of the Vietnamese Mathematical Olympiad 1999 Category B, given [2002 : 7–8].

1. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence defined by

$$u_1 \;=\; 1, \quad u_2 \;=\; 2 \quad \text{and} \quad u_{n+2} \;=\; 3u_{n+1} - u_n$$

for all $n = 1, 2, \ldots$. Prove that

$$u_{n+2} + u_n \;\geq\; 2 + \frac{u_{n+1}^2}{u_n}$$

for all $n = 1, 2, \ldots$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Vedula N. Murty, Dover, PA, USA. We give Bataille's write-up.

Let $\{f_n\}$ be the usual Fibonacci sequence, defined by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all positive integers n. Clearly, $u_1 = f_1$ and $u_2 = f_3$. Suppose that $u_k = f_{2k-1}$ and $u_{k+1} = f_{2k+1}$ for some integer $k \geq 1$. Then

$$\begin{aligned} u_{k+2} &= 3u_{k+1} - u_k \;=\; 3f_{2k+1} - f_{2k-1} \\ &= 2f_{2k+1} + f_{2k} \;=\; f_{2k+1} + f_{2k+2} \;=\; f_{2k+3}\,. \end{aligned}$$

Thus, $u_n = f_{2n-1}$ for all $n \geq 1$ (by induction).

Now, for any positive integer n,

$$\begin{aligned} u_{n+1}^2 - u_n u_{n+2} &= f_{2n+1}^2 - f_{2n-1}f_{2n+3} \\ &= f_{2n+1}^2 - f_{2n-1}f_{2n+1} - f_{2n-1}f_{2n+2} \\ &= f_{2n+1}f_{2n} - f_{2n-1}f_{2n+2} \\ &= f_{2n}^2 + f_{2n-1}f_{2n} - f_{2n-1}f_{2n+2} \\ &= f_{2n}^2 - f_{2n-1}f_{2n+1} \;=\; (-1)^{2n+1} = -1\,, \end{aligned}$$

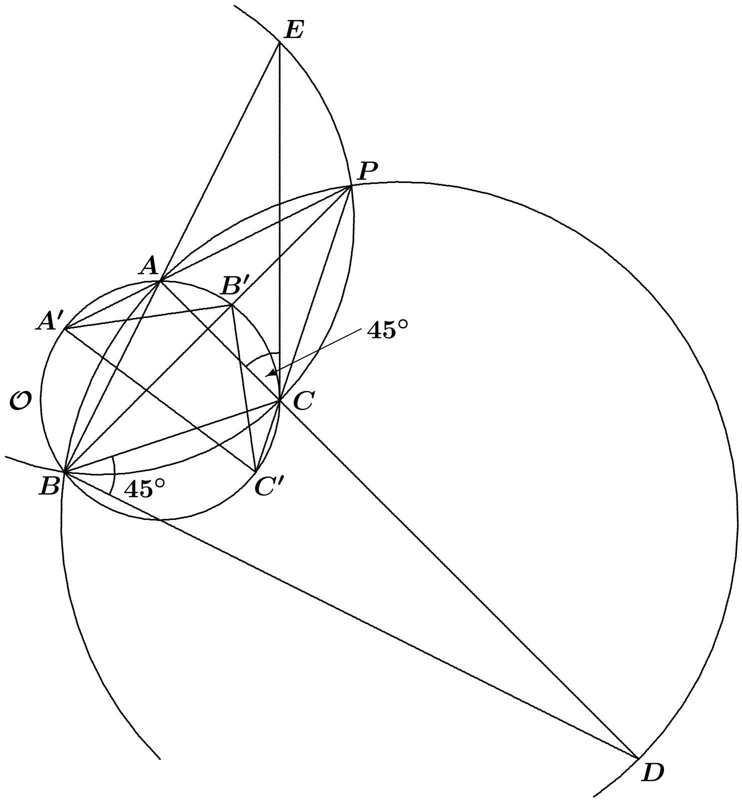
where we have used the well-known relation $f_m^2 - f_{m-1}f_{m+1} = (-1)^{m+1}$, valid for all $m \geq 2$. As a result,

$$u_{n+2} + u_n - \frac{u_{n+1}^2}{u_n} \;=\; u_n + \frac{1}{u_n}\,.$$

The desired inequality follows, since $x + \frac{1}{x} \geq 2$ for all positive x.

2. Let ABC be a triangle inscribed in the circle $\mathcal{O}$. Locate the position of the points P, not lying in the circle $\mathcal{O}$, of the plane (ABC) with the property that the lines PA, PB, PC intersect the circle $\mathcal{O}$ again at points A', B', C' such that $A'B'C'$ is a right-angled isosceles triangle with $\angle A'B'C' = 90^\circ$.

Solved by Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



In the following solution we assume that $\angle A > 45^\circ$ and $\angle C > 45^\circ$. But the following construction and proof work in other cases with minor changes.

Let D be the point on AC produced beyond C such that $\angle CBD = 45^\circ$, and let E be the point on AB produced beyond A such that $\angle ACE = 45^\circ$. Let P be the intersection of the circumcircles of $\triangle ABD$ and $\triangle BCE$ other than B.

Claim. Then P is the point we are looking for.

Proof. Let A', B', and C' be the points of intersection of PA, PB, and PC with circle O other than A, B, and C. Then

$$\begin{aligned} \angle B'A'C' &= \angle B'CP = \angle BB'C - \angle BPC \\ &= \angle BAC - \angle BEC = \angle ACE = 45^\circ, \\ \text{and} \quad \angle B'C'A' &= \angle B'AP = \angle AB'B - \angle APB \\ &= \angle ACB - \angle ADB = \angle CBD = 45^\circ. \end{aligned}$$

Hence, $\triangle A'B'C'$ is a right-angled isosceles triangle with $\angle A'B'C' = 90^\circ$.

3. Consider real numbers a, b such that all roots of the equation

$$ax^3 - x^2 + bx - 1 \;=\; 0$$

are real and positive.

Determine the smallest possible value of the following expression:

$$P \;=\; \frac{5a^2 - 3ab + 2}{a^2(b-a)}\,.$$

Solution by Vedula N. Murty, Dover, PA, USA.

Let r_1, r_2, r_3 be the roots of the given equation. Then

$$\begin{aligned} r_1 + r_2 + r_3 &= \frac{1}{a}\,, & (1)\\ r_1r_2 + r_2r_3 + r_3r_1 &= \frac{b}{a}\,, & (2)\\ r_1r_2r_3 &= \frac{1}{a}\,. & (3) \end{aligned}$$

Suppose r_1, r_2, r_3 are real and positive. This implies that $a > 0$ and $b > 0$. Using the AM–GM Inequality with equations (1) and (3), we obtain

$$\frac{1}{a} \;\geq\; 3\sqrt{3}\,. \qquad (4)$$

Using the inequality $3(r_1r_2+r_2r_3+r_3r_1) \leq (r_1+r_2+r_3)^2$ with equations (1) and (2), we obtain $3ab \leq 1$, and hence,

$$P \;=\; \frac{5a^2 - 3ab + 2}{a^2(b-a)} \;\geq\; \frac{5a^2+1}{a^2(b-a)}\,.$$

The inequality $(r_2+r_3-r_1)(r_3+r_1-r_2)(r_1+r_2-r_3) \leq r_1r_2r_3$ gives us

$$\left(\frac{1}{a} - 2r_1\right)\left(\frac{1}{a} - 2r_2\right)\left(\frac{1}{a} - 2r_3\right) \;\leq\; \frac{1}{a}\,.$$

This, with equations (1), (2), and (3), yields $9a^2 - 4ab + 1 \geq 0$; that is,

$$5a^2 + 1 \;\geq\; 4a(b-a)\,.$$

Then

$$P \;\geq\; \frac{4a(b-a)}{a^2(b-a)} \;=\; \frac{4}{a} \;\geq\; 12\sqrt{3}\,,$$

where the last step follows by (4).

Equality is attained when $a = \frac{\sqrt{3}}{9}$ and $b = \sqrt{3}$ (in which case the original equation is $\left(\frac{x}{\sqrt{3}} - 1\right)^3 = 0$). We conclude that the smallest value of P is $12\sqrt{3}$.

4. Let $f(x)$ be a continuous function defined on $[0,1]$ such that

(i) $f(0) = f(1) = 0$,

(ii) $2f(x) + f(y) = 3f\left(\frac{2x+y}{3}\right) \quad \forall\, x, y \in [0,1]$.

Prove that $f(x) = 0$ for all $x \in [0,1]$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Lefkogia, Crete, Greece. We give Bataille's solution, modified slightly by the editor.

Since $|f|$ is continuous on the closed and bounded interval $[0,1]$, there exists a real number $a \in [0,1]$ at which $|f|$ attains its maximum M; that is, $|f(x)| \le |f(a)| = M$ for all $x \in [0,1]$. We will show that $M = 0$, which implies that f is the zero function.

Case 1: $0 \le a \le \frac{1}{2}$.

Let $a_1 = 2a/3$ and $b_1 = 5a/3$. Then $0 \le a_1 \le a \le b_1 < 1$ and $a = \dfrac{2a_1 + b_1}{3}$. Hence, using (ii),

$$\begin{aligned} M &= |f(a)| = \left|f\left(\frac{2a_1+b_1}{3}\right)\right| = \left|\tfrac{2}{3}f(a_1) + \tfrac{1}{3}f(b_1)\right| \\ &\le \tfrac{2}{3}|f(a_1)| + \tfrac{1}{3}|f(b_1)| \le \tfrac{2}{3}M + \tfrac{1}{3}M = M\,, \end{aligned}$$

from which we deduce that $|f(a_1)| = M$.

Iterating, we construct a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n = (\frac{2}{3})^n a$ and $|f(a_n)| = M$ for all positive integers n. Since $\{a_n\}$ converges to 0, the continuity of f implies that $M = \lim\limits_{n\to\infty} |f(a_n)| = |f(0)| = 0$.

Case 2: $\frac{1}{2} < a \le 1$.

Let $g(x) = f(1-x)$. Then g satisfies all the given conditions on f, and the maximum of $|g|$ is M, attained at $1-a$. Since $0 \le 1-a < \frac{1}{2}$, we may apply Case 1 to the function g to deduce that $M = 0$.

5. The base side and the altitude of a regular hexagonal prism $ABCDEF$, $A'B'C'D'E'F'$ are equal to a and h, respectively. Prove that six planes $(AB'F)$, $(CD'B)$, $(EF'D)$, $(D'EC)$, $(F'AE)$ and $(B'CA)$ are tangent to the same sphere. Determine the centre and the radius of this sphere.

Solution by Christopher J. Bradley, Bristol, UK, modified by the editor.

First, note the misprint in the question: the second of the six planes should be $(CD'B)$, not $(CD'B')$. [*Ed.* This has been corrected already in the problem statement above.]

We introduce rectangular coordinates (x, y, z) such that the hexagons $ABCDEF$ and $A'B'C'D'E'F'$ lie in the planes $z = 0$ and $z = h$, respectively, and the coordinates of A, B, C, D, E, F are as follows:

$$A = a\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \quad B = a\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \quad C = a(1,0,0)\,,$$

$$D = a\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right), \quad E = a\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right), \quad F = a(-1,0,0).$$

(The coordinates of A', B', C', D', E', F' are the same except that their z-coordinates have the value h instead of 0.)

Note that the three planes $(AB'F)$, $(CD'B)$, and $(EF'D)$ are placed symmetrically around the prism under a rotation of 120°, as are the three planes $(D'EC)$, $(F'AE)$, and $(B'CA)$. Therefore, if there is a sphere with centre on the z-axis which is tangent to the planes $(AB'F)$ and $(D'EC)$, then this sphere will be tangent to all six planes.

The planes $(AB'F)$ and $(D'EC)$ have the respective equations

$$\frac{x}{a} - \frac{1}{\sqrt{3}}\frac{y}{a} - \frac{z}{h} + 1 = 0 \quad \text{and} \quad \frac{x}{a} - \sqrt{3}\,\frac{y}{a} - \frac{z}{h} - 1 = 0.$$

We now look for a point $(0,0,k)$ which is the same distance r from these two planes. This condition gives us

$$r = \frac{a}{\mu}\left|1 - \frac{k}{h}\right| = \frac{a}{\nu}\left|1 + \frac{k}{h}\right|,$$

where

$$\mu = \sqrt{\frac{4}{3} + \left(\frac{a}{h}\right)^2} \quad \text{and} \quad \nu = \sqrt{4 + \left(\frac{a}{h}\right)^2}.$$

Solving for k, we obtain two solutions, with corresponding values for r:

$$k = \left(\frac{\nu-\mu}{\nu+\mu}\right)h = \frac{3}{8}(\nu-\mu)^2 h, \qquad r = \frac{3}{4}a(\nu-\mu);$$

$$k = \left(\frac{\nu+\mu}{\nu-\mu}\right)h = \frac{3}{8}(\nu+\mu)^2 h, \qquad r = \frac{3}{4}a(\nu+\mu).$$

We have found *two* spheres that meet the requirements in the problem. Their centres are on the axis of symmetry of the prism at $(0,0,k)$, where the two values of k and the corresponding values of the radius r are given by the equations above.

6. Two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are determined recursively by

$$x_1 = 1, \qquad y_1 = 2, \qquad \text{and}$$
$$x_{n+1} = 22y_n - 15x_n,$$
$$y_{n+1} = 17y_n - 12x_n,$$

for all $n = 1, 2, \ldots$.

(a) Prove that

(i) $\{x_n\}$ and $\{y_n\}$ are not equal to zero for all $n = 1, 2, \ldots$.

(ii) The sequences $\{x_n\}$ and $\{y_n\}$ contain infinitely many positive terms and infinitely many negative terms.

(b) Are the $(1999^{1945})^{\text{th}}$ terms of the sequence $\{x_n\}$ and the sequence $\{y_n\}$ divisible by 7 or not?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution.

(a) (i) From the recurrence relations, we get $17x_{n+1} - 22y_{n+1} = 9x_n$, for all $n = 1, 2, 3, \ldots$. Then, for all $n = 2, 3, \ldots$,

$$x_{n+1} \;=\; 22y_n - 15x_n \;=\; 2x_n - 9x_{n-1}\,. \tag{1}$$

Similarly,

$$y_{n+1} \;=\; 2y_n - 9y_{n-1}\,. \tag{2}$$

Note that $x_1 = 1$, $x_2 = 29$, and $y_1 = 2$, $y_2 = 22$. We deduce easily by induction that x_n is odd and $y_n \equiv 2 \bmod 4$, for all $n = 1, 2, \ldots$. It follows that x_n and y_n are not equal to 0.

(ii) From (1), for all $n = 2, 3, \ldots$, we have

$$x_{n+2} \;=\; -5x_n - 18x_{n-1}\,.$$

If x_n and x_{n-1} have the same sign, then x_{n+2} and x_n have opposite signs. Thus, in every four consecutive terms of the sequence $\{x_n\}$, there are always two which have opposite signs. It follows that the sequence $\{x_n\}$ contains infinitely many positive terms and infinitely many negative terms.

We prove the same result in the same way for the sequence $\{y_n\}$.

(b) From (1), we have $x_{n+1} \equiv 2(x_n - x_{n-1}) \pmod 7$, for all $n \geq 2$. Suppose that, for some $n \geq 2$, we have $x_n \equiv x_{n-1} \not\equiv 0 \pmod 7$. Then $x_{n+1} \equiv 0 \pmod 7$, $x_{n+2} \equiv 5x_n \not\equiv 0 \pmod 7$, $x_{n+3} \equiv 3x_n \not\equiv 0 \pmod 7$, and $x_{n+4} \equiv 3x_n \not\equiv 0 \pmod 7$. Since, $x_1 \equiv x_2 \equiv 1 \pmod 7$, we deduce by induction that $x_n \equiv 0 \pmod 7$ if and only if $n \equiv 3 \pmod 4$.

Now, since $3^2 \equiv 1 \pmod 4$, we have

$$1999^{1945} \;\equiv\; 3^{1945} \;\equiv\; 3 \pmod 4\,.$$

Thus, x_n is divisible by 7 when $n \equiv 1999^{1945}$.

From (2), we have $y_{n+1} \equiv 2(y_n - y_{n-1}) \pmod 7$, for all $n \geq 2$. Suppose $y_{n+1} \equiv 0 \pmod 7$ for some $n \geq 4$. Then, $y_n \equiv y_{n-1} \pmod 7$. Since $y_n \equiv 2(y_{n-1} - y_{n-2}) \pmod 7$, we get $y_{n-1} \equiv 2y_{n-2} \pmod 7$. Then, since $y_{n-1} \equiv 2(y_{n-2} - y_{n-3}) \pmod 7$, we deduce that $y_{n-3} \equiv 0 \pmod 7$. Thus, if $y_{n+1} \equiv 0 \pmod 7$, then $y_{n-3} \equiv 0 \pmod 7$.

By induction, it follows that at least one of the first five terms of the sequence (y_n) must be divisible by 7. But it is easy to verify that this is not the case.

Thus, $y_n \not\equiv 0 \pmod 7$ for all $n \geq 1$. In particular, $y_n \not\equiv 0 \pmod 7$ for $n = 1999^{1945}$.

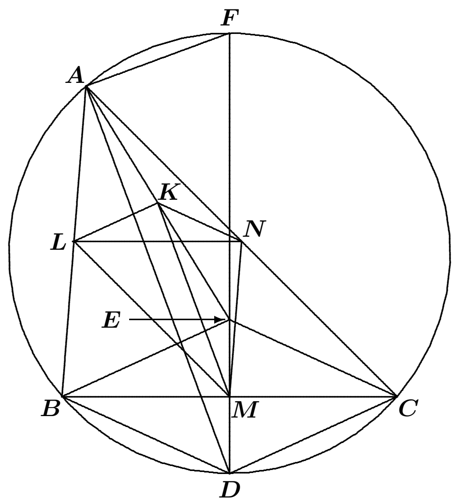
Next we turn to solutions to problems of the 16th Balkan Mathematical Olympiad, given [2002 : 8].

1. Given an acute-angled triangle ABC, let D be the mid-point of the arc BC of the circumcircle of ABC not containing A. The points which are symmetric to D with respect to the line BC and the centre of the circumcircle are denoted by E and F, respectively. Finally, let K stand for the mid-point of $[EA]$. Prove that:

(a) the circle passing through the mid-points of the edges of the triangle ABC, also passes through K;

(b) the line passing through K and the mid-point of $[BC]$ is perpendicular to AF.

Solved by Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's account.



(a) Letting L, M, and N be the mid-points of AB, BC, and CA, respectively, we have $LM \parallel AC$ and $MN \parallel AB$. Thus,

$$\angle LMN = \angle LAN = \angle BAC .$$

Since K is the mid-point of AE, we see that $LK \parallel BE$ and $KN \parallel EC$. Thus, $\angle LKN = \angle BEC$. Since E is the reflection of D with respect to BC, we have $\angle BEC = \angle BDC$, and then $\angle LKN = \angle BDC$. Hence,

$$\angle LKN + \angle LMN = \angle BDC + \angle BAC = 180^\circ .$$

Therefore, L, M, N, and K are concyclic.

(b) Since D is the mid-point of the arc BC, we have $BD = DC$. Thus, $BDCE$ is a rhombus, and M is the mid-point of DE. Since K and M are mid-points of AE and DE, respectively, we get $KM \parallel AD$.

Since F is symmetric to D with respect to the centre of the circumcircle of $\triangle ABC$, we see that DF is a diameter of this circle. Thus, $\angle DAF = 90°$; that is, $AD \perp AF$. Since $KM \parallel AD$, we obtain $KM \perp AF$.

2. Let $p > 2$ be a prime number such that 3 divides $p - 2$. Let

$$S = \{y^2 - x^3 - 1 \mid x, y \text{ are integers, } 0 \leq x, y \leq p - 1\}.$$

Prove that at most $p - 1$ elements of the set S are divisible by p.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bradley's write-up.

Since p is a prime of the form $3k + 2$, we have $k^{3k+1} \equiv 1 \pmod{p}$ for $k \not\equiv 0 \pmod{p}$, by Fermat's Theorem. Since $3 \nmid (3k + 1)$, we see that $h^3 \equiv 1 \pmod{p}$ implies that $h \equiv 1 \pmod{p}$. Suppose now that $s^3 \equiv t^3 \pmod{p}$ and $s, t \not\equiv 0 \pmod{p}$. Then $(t^{-1}s)^3 \equiv 1 \pmod{p}$ implies that $t \equiv s \pmod{p}$. It follows that the cubic residues modulo $3k + 2$ are $1, 2, 3, \ldots, p - 1$. We also have $0^3 \equiv 0 \pmod{p}$. Hence, $x^3 \equiv 0, 1, 2, 3, \ldots, p - 1 \pmod{p}$ once each as x ranges from 0 to $p - 1$.

Then, for each value of $y^2 - 1$ where $0 \leq y \leq p - 1$, there is one and only one value of x^3 such that $y^2 - x^3 - 1$ is divisible by p. At first sight it appears as though there may be as many as p elements of S divisible by p. However, at least two of them, namely $1^2 - 0^3 - 1$ and $3^2 - 2^3 - 1$, are equal to 0. Hence, there are at most $p - 1$ elements of s that are divisible by p.

3. Let ABC be an acute-angled triangle; M, N and P are the feet of the perpendiculars from the centroid G of the triangle upon its sides AB, BC and CA respectively. Prove that

$$\frac{4}{27} < \frac{\text{area}(MNP)}{\text{area}(ABC)} \leq \frac{1}{4}.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and by Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bataille's solution.

As usual, let $a = BC$, $b = CA$, and $c = AB$, and let $[XYZ]$ denote the area of $\triangle XYZ$. Let $\rho = \dfrac{[MNP]}{[ABC]}$. We will prove that $\frac{2}{9} < \rho \leq \frac{1}{4}$, which is slightly stronger than requested.

First recall that $[GBC] = [GAC] = [GAB] = \frac{1}{3}[ABC]$. This follows at once be remarking that, for instance, GN is a third of the altitude from A in $\triangle ABC$. Now, since $\angle MGN = 180° - \angle B$,

$$\begin{aligned} 2[GMN] &= GM \cdot GN \cdot \sin B \\ &= \frac{2[GAB]}{c} \cdot \frac{2[GBC]}{a} \sin B = \frac{4}{9} \frac{[ABC]^2}{ac} \sin B. \end{aligned}$$

Similar results hold for $[GNP]$ and $[GMP]$. Thus,

$$[MNP] = \frac{2[ABC]^2}{9}\left(\frac{\sin A}{bc} + \frac{\sin B}{ca} + \frac{\sin C}{ab}\right).$$

Since $[ABC] = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$, it follows that

$$\rho = \tfrac{1}{9}(\sin^2 A + \sin^2 B + \sin^2 C).$$

By the usual trigonometric formulas, we get

$$\begin{aligned}\sin^2 A + \sin^2 B + \sin^2 C &= \tfrac{3}{2} - \tfrac{1}{2}(\cos 2A + \cos 2B + \cos 2C) \\ &= 2(1 + \cos A\cos B\cos C).\end{aligned}$$

[*Ed.* This calculation is given in more detail in the solution to problem 2676 [2002 : 475].] Since $\triangle ABC$ is acute-angled, $\cos A\cos B\cos C > 0$, and hence $\rho > \frac{2}{9}$. Furthermore, by the Law of Cosines,

$$\cos A\cos B\cos C = \frac{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}{8a^2b^2c^2}.$$

By the AM–GM Inequality,

$$\begin{aligned}\sqrt{(b^2+c^2-a^2)(c^2+a^2-b^2)} &\le \tfrac{1}{2}(2c^2) = c^2, \\ \sqrt{(c^2+a^2-b^2)(a^2+b^2-c^2)} &\le a^2, \\ \sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)} &\le b^2.\end{aligned}$$

Multiplying these inequalities gives

$$(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2) \le a^2b^2c^2,$$

which implies that $\cos A\cos B\cos C \le \frac{1}{8}$. The inequality $\rho \le \frac{1}{4}$ follows immediately.

Klamkin comments on the inequalities $2 < \sin^2 A + \sin^2 B + \sin^2 C \le \frac{9}{4}$, *which Bataille has proved above:*

The upper bound of $\frac{9}{4}$ is known to hold for all triangles, with equality if and only if the triangle is equilateral (see [1]). The lower bound of 2 is known to be the best possible for acute triangles (again, see [1]). One can get arbitrarily close to this lower bound for triangles of angles 2ε, $90-\varepsilon$, $90-\varepsilon$ where ε is arbitrarily small.

Reference:

[1] O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović & P.M. Vasić, *Geometric Inequalities*, Groningen, 1969, p. 18.

That completes this number of the *Corner*. Send me your nice solutions and generalizations.