The Birthday Problem Revisited

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Problems about birthday probabilities are among the most interesting problems in probability due to their surprising answers, and they are more germane to the beginning probability student than many of the other problems in elementary probability texts. We will discuss three of the many birthday problems in existence.

In these problems, February 29$^{\text{th}}$ is ignored as a possible birthday and the other 365 days are treated as equally likely.

The Classical Birthday Problem. *What is the minimum number of people so that the probability of two or more of them having the same birthday exceeds one half?*

We need to find the probability that, among n people, at least two have a common birthday. We will solve the complementary problem of finding the probability that *no two people have the same birthday*.

Since each person has 365 possible birthdays, the number of possible ways for n people to have birthdays is 365^n. The number of ways for n people to have no matching birthdays is

$$365 \times 364 \times \cdots \times (365 - n + 1)\,,$$

because there are 365 possible birthdays for the first person, 364 possible *different* birthdays for the second person, 363 for the third person, and so on. The last person can have a birthday different from the first $(n-1)$ people in $365 - (n - 1)$ ways. Thus, the probability that no two of the n people have the same birthday is

$$\frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}\,.$$

Now the probability P_n of at least one matching pair of birthdays among the n people is

$$P_n \;=\; 1 - \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}\,.$$

According to our objective, we have to find the smallest n such that $P_n > \frac{1}{2}$. The following table shows that the answer is 23.

n	5	10	15	20	21	22	23	24	25
P_n	0.027	0.117	0.253	0.411	0.444	0.476	0.507	0.538	0.569

The Birthmate Problem. *What is the smallest number of strangers you need to meet in order to have at least a 50% chance of finding one whose birthday is the same as yours?*

This time, we need to find the probability that, among n strangers, at least one has the same birthday as yours. We again consider the complementary problem of finding the probability that no birthdays are the same as yours.

First of all, the probability that some birthday is different from yours is $\frac{364}{365}$, or $\frac{N-1}{N}$, if we set $N = 365$. Since we have n strangers, the probability that nobody among them has your birthday is $\left(\frac{N-1}{N}\right)^n$. Then the probability P_n that at least one person's birthday matches yours is

$$P_n = 1 - \left(\frac{N-1}{N}\right)^n = 1 - \left(1 - \frac{1}{N}\right)^n .$$

As N increases, the value of $\left(1 - \frac{1}{N}\right)^N$ approaches e^{-1}. We therefore have the approximation

$$\left(1 - \frac{1}{N}\right)^N \approx e^{-1} .$$

Taking the N^{th} root of both sides, we obtain $1 - \frac{1}{N} \approx e^{-\frac{1}{N}}$. Now,

$$\left(1 - \frac{1}{N}\right)^n \approx \left(e^{-\frac{1}{N}}\right)^n = e^{-\frac{n}{N}} .$$

Hence, $P_n \approx 1 - e^{-\frac{n}{N}}$.

We want $P_n \geq \frac{1}{2}$, which implies $e^{-\frac{n}{N}} \leq \frac{1}{2}$. Taking natural logarithms on both sides, we have

$$-\frac{n}{N} \approx \ln\left(\tfrac{1}{2}\right) \approx -0.693 ,$$

or $n \approx 0.693N$. Since $N = 365$, we have

$$n \approx 252.945 .$$

Since n must be an integer, it must be 253. Thus, we need to ask at least 253 strangers to have at least a 50% chance that one of their birthdays will coincide with our own.

Birthmate Problem for a Group. *If n people meet by chance, what is the probability that they all have the same birthday?*

Observe that the probability that any given day is the birthday for a given person is $\frac{1}{365}$. Therefore, the probability that n persons all have a particular birthday is $\frac{1}{365^n}$. Now, adding over all 365 possible birthdays, the probability that all n persons have the same birthday is

$$365\left(\frac{1}{365^n}\right) = \frac{1}{365^{n-1}}.$$

To this point we have assumed that we were ignoring leap years. This allowed us to simplify the calculations. Readers may wish to reconsider the previous examples without the assumption. In this final example, if we were working with a leap year instead, the probability would change to $\frac{1}{366^{n-1}}$. In order to consider both possibilities, we look at a 4–year cycle, in which case the probability can be seen to be $(1 + 365 \times 4^n)/1461^n$. All three of these probabilities are computed in the following table for selected values of n.

n	2	10	20
non-leap year	0.002740	8.697×10^{-24}	2.072×10^{-49}
leap year	0.002732	8.485×10^{-24}	1.967×10^{-49}
overall	0.002736	8.638×10^{-24}	2.044×10^{-49}

Therefore, for all n people to have the same birthday is very unlikely.

Bibliography

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