Pólya's Paragon

What's the difference?

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"Little Johnnie encounters the following list of numbers 12, 49, 62, 57, 40, 17, ... *What is the next number in the list?"*

Questions like this have been seen in mathematics text books and on mathematics contests over the years. Is there a general way to attack them to find the next few terms?

It turns out that there is a powerful method called *finite differences* that deals with sequences quite well. The main idea is to find the differences between consecutive terms and look for a pattern. Many mathematical functions reveal their structure under this method. Let's see a couple of examples in action.

Example 1: Find the next term in the sequence 1, 3, 5, 7, 9, ...

Solution: This example is trivial, but it reveals our general technique. If we call the terms t_1, t_2, t_3, t_4, t_5, ... and the differences $d_1 = t_2 - t_1$, $d_2 = t_3 - t_2$, etc., we get the following table:

t_n	1		3		5		7		9
d_n		2		2		2		2	

In case we didn't see the pattern in the original sequence, the sequence of differences is "easier". We can produce the next term in the original sequence by realizing that $t_6 - t_5 = d_5 = 2$ (since all the differences are 2), and therefore,

$$t_6 = 2 + t_5 = 2 + 9 = 11 .$$

Technically, the differences that we produced are called *first differences*. If we rename our differences ${}_1d_1$, ${}_1d_2$, ${}_1d_3$, ${}_1d_4$, ${}_1d_5$, ..., then we can define the *second differences* as ${}_2d_1 = {}_1d_2 - {}_1d_1$, ${}_2d_2 = {}_1d_3 - {}_1d_2$, etc. These will be useful in the next problem.

Example 2: Find the next term in the sequence 1, 4, 9, 16, 25, ...

Solution: Again, we have a sequence that is easy to recognize. This time the first differences are not the same; hence, we will continue to the second differences.

t_n	1		4		9		16		25
${}_1d_n$		3		5		7		9	
${}_2d_n$			2		2		2		

If we didn't see a pattern in the first differences, we can easily see one in the second differences. The next second difference must be ${}_2d_4 = 2$, which gives the next first difference as ${}_1d_5 = 9 + 2 = 11$, and then the next term of the sequence is $t_6 = 25 + 11 = 36$.

At this time a pattern is emerging. Notice that when the sequence can be modelled by a linear function, the first differences are constant. Similarly, when the sequence can be modelled by a quadratic, the second differences are constant. This pattern continues:

Theorem. *If a sequence $\{t_n\}$ can be modelled by a polynomial of degree k, then the k^{th} differences are constant.*

We can see the proof of this by looking at a lemma first.

Lemma. *If a sequence $\{t_n\}$ can be modelled by a polynomial of degree k, then the first differences $\{{}_1d_n\}$ can be modelled by a polynomial of degree $k - 1$.*

Proof: We will only sketch the main idea (try to construct your own proof). Suppose we have a sequence $\{t_n\}$ where $t_n = n^k$. Then

$${}_1d_n \;=\; t_{n+1} - t_n \;=\; (n+1)^k - n^k\,.$$

Using the Binomial Theorem, we can see that ${}_1d_n$ is of degree $k - 1$.

Using the lemma, you can prove the theorem by induction. With a little experimentation, you will also see that the difference that is constant can be used to determine the coefficient of the highest degree term. See if you can discover the connection, and provide a proof. With this new theorem, you should be set to attack the original sequence.

We will return to this topic next issue and expand on it. Here are some problems to keep you busy in the meantime (homework, one might say).

Examine the differences for the following sequences, and see if you can make any predictions.

1. $a_n = 2^n$.
2. $b_n = 5^n$.
3. $c_n = 2 \times 3^n$.
4. $1, 1, 2, 3, 5, 8, 13, \ldots$. (This is the *Fibonacci sequence*. Each term is the sum of the previous two terms.)

It should be noted, that any sequence like the one with which we started can be continued in **any** *way. That is, we can pick any number to go next and find a polynomial that will match those numbers. See the article on Lagrange Interpolation from the Skoliad Corner* [2001 : 386–388].