Mayhem Solutions

M69. *Proposed by the Mayhem Staff.*

A sequence of digits is formed by writing the digits from the natural numbers in the order that they appear. The sequence starts:

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 1,\ 0,\ 1,\ 1,\ 1,\ 2,\ \ldots .$$

What is the 2002^{nd} digit in the sequence?

Solution by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia.

Writing the first 9 natural numbers requires 9 digits. Writing the next 90 numbers requires $90 \times 2 = 180$ digits. Thus, to write the numbers from 1 to 99 requires 189 digits. The next 9 groups of 100 natural numbers each require $100 \times 3 = 300$ digits. Clearly, only 6 of these groups are required. Once we have written the digits of the number 699, we will have used 1989 digits. Now, $2002 - 1989 = 13$ more digits are required. Therefore, the 2002^{nd} digit is the '7' from 704.

M70. *Proposed by the Mayhem Staff.*

What is the smallest positive multiple of 15 that is made up of only the digits 0, 4, and 7, each appearing the same number of times?

Solution by the Mayhem Staff.

Let n be the required positive integer. Clearly, $15|n$ implies both $5|n$ and $3|n$. Since $5|n$ and the only digits in n are 0, 4, and 7, it follows that n ends in the digit 0. Since $3|n$, the digital sum of n must be a multiple of 3, and thus, the number of repeats of each digit must be a multiple of 3. Since n is the *smallest* multiple of 15, we try the number of repeats equal to 3. Again, since we want the smallest positive integer, we need to have the smaller digits at the beginning of the integer. Thus, since leading zeroes are not allowed, $n = 400447770$.

M71. *Proposed by Richard Hoshino, Dalhousie University, Halifax, NS.*

Let $x = a + b - c$, $y = a + c - b$ and $z = b + c - a$, where a, b and c are prime numbers. Given that $x^2 = y$ and $\sqrt{z} - \sqrt{y}$ is the square of a prime number, determine all possible values for the product abc.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

The unique solution of the simultaneous linear equations $a+b-c = x$, $a + c - b = y$, and $b + c - a = z$ is

$$(a, b, c) \ = \ \left(\tfrac{1}{2}(x + y), \tfrac{1}{2}(x + z), \tfrac{1}{2}(y + z)\right) .$$

Setting $y = x^2$, we have

$$\begin{aligned} a &= \tfrac{1}{2}(x + x^2)\,, & (1)\\ b &= \tfrac{1}{2}(x + z)\,, & (2)\\ c &= \tfrac{1}{2}(x^2 + z)\,. & (3) \end{aligned}$$

From (1), we have $x = \dfrac{-1 \pm \sqrt{1+8a}}{2}$. Since x is an integer, we get $1 + 8a = T^2$, for some odd positive integer T. Hence, $2a = \dfrac{T-1}{2} \cdot \dfrac{T+1}{2}$. Therefore, since a is prime, we have $\dfrac{T-1}{2} = 2$ and $\dfrac{T+1}{2} = a$, from which we conclude that $T = 5$ and $a = 3$. Then, using (1), $x = 2$ or $x = -3$.

If $x = 2$, then $y = 4$ and $\sqrt{z} - 2 = p^2$, for some prime p. Thus, $z = \left(p^2 + 2\right)^2$. From (2), we see that z is even. Hence, $p = 2$ and $z = 36$. Substituting into (3), we get $c = 20$, which contradicts the fact that c is prime.

Therefore, $x = -3$. Then $y = 9$ and $z = \left(p^2 + 3\right)^2$, for some prime p. Now, (2) and (3) imply that z is odd. This happens only if $p = 2$, which gives $z = 49$. Then $b = 23$, and $c = 29$.

Then $abc = (3)(23)(29) = 2001$.

M72. *Proposed by J. Walter Lynch, Athens, GA, USA.*

You have a cup of coffee and a cup of tea. The cups are identical and each contains the same amount of liquid as the other. You take a teaspoon full of coffee out of the coffee cup and put it into the teacup. You then take a teaspoon full of the mixture out of the teacup and put it into the coffee cup. Which is greater, the amount of coffee in the teacup, the amount of tea in the coffee cup, or are they the same?

Solution by the Mayhem Staff.

Exactly 1 teaspoon of liquid was transferred in each direction; thus, the volume of liquid in each cup is the same both before and after the exchange. Suppose the amount of coffee in the teacup is greater than the amount of tea in the coffee cup. Then, since the total amount of coffee equals the total amount of tea, the remaining amount of coffee in the coffee cup is less than the remaining amount of tea in the teacup. This implies that the amount of liquid in the coffee cup is less than that in the teacup, which contradicts the statement in the first sentence.

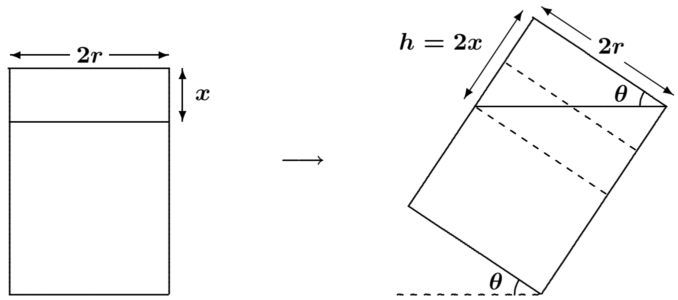
Similarly the supposition that the amount of coffee in the teacup is less than the amount of tea in the coffee cup leads to a contradiction.

Thus, these two amounts must be equal.

M73. *Proposed by J. Walter Lynch, Athens, GA, USA.*

A right circular cylinder with radius r and height h contains a liquid to within x of the top of the cylinder. Find the angle through which the cylinder must be tilted in order for the liquid to start to pour out. (Assume that there is enough liquid in the cylinder so that the surface of the liquid does not intersect the bottom of the cylinder before the liquid starts to pour out.)

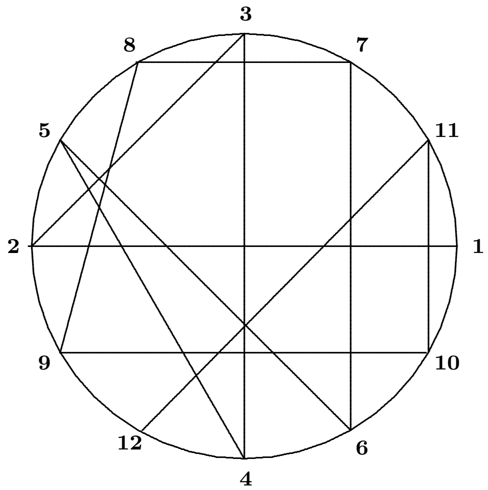
Solution by the Mayhem Staff.



As the cylinder is tilted, the volume of the unfilled space inside the cylinder remains constant as long as no liquid spills over. Consider the cylinder tilted to the point where liquid is just starting to spill (as in the figure above). Let θ denote the angle of tilt at this point. The liquid surface is now an ellipse. Draw a circle around the cylinder from the point on this ellipse which is diametrically opposite the lip of the cylinder. This circle marks off a cylinder of height h at the top of the original cylinder. The surface ellipse of the liquid divides this second cylinder into two congruent 'wedges', one of which is the unfilled space. Therefore, the volume of the entire second cylinder is twice that of the unfilled space. It follows that $h = 2x$. Hence, the angle θ is given by $\tan\theta = \frac{2x}{2r} = \frac{x}{r}$; that is, $\theta = \arctan\left(\frac{x}{r}\right)$.

M74. *Proposed by the Mayhem Staff.*

A circle has 12 equally spaced points placed on its circumference. How many ways can the numbers 1 through 12 be assigned to the points so that if the points 1 through 12 are connected with line segments, in order, the segments do not cross? An example of a **bad** arrangement is illustrated below.



Solution by the Mayhem Staff.

We will use the fact that there is at least one chord drawn to or from every point. We can place the label '1' at any of the 12 points. However, the label '2' is now forced to go at one of the two points adjacent to the point labelled '1', since, if we skip past an adjacent point, a chord to or from this adjacent point will intersect the chord joining the points that we have labelled '1' and '2'. Similarly, the label '3' can only be placed at one of the two points adjacent to the '1–2' block. There is a choice of two possible points for each successive label until the label '12', for which there is only one remaining point. Thus, there are $12\left(2^{10}\right) = 12\,288$ labellings.

M75. *Proposed by the Mayhem Staff.*

The increasing sequence 1, 5, 6, 25, 26, 30, 31, 125, 126, ... consists of positive integers that can be formed by adding distinct powers of 5. What is the 75^{th} integer in the sequence?

Solution by the Mayhem Staff.

We write the sequence in base 5 as 1, 10, 11, 100, 101, 110, ... If, instead of considering this sequence in base 5, we now consider it as a sequence of binary numbers, we note that the number in position n is n. Thus, since 75 may be written in binary as 1001011, the required number is $5^6 + 5^3 + 5 + 1 = 15\,756$.

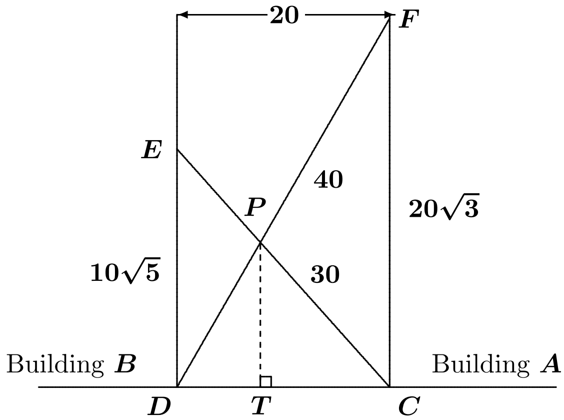
M76. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Two buildings A and B are twenty feet apart. A ladder thirty feet long has its lower end at the base of building A and its upper end against building B. Another ladder forty feet long has its lower end at the base of building B and its upper end against building A.

How high above the ground is the point where the ladders intersect?

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let CE and DF be the ladders, as shown in the diagram below. Let P be the point of intersection of the ladders, and let T be the point on the ground directly below P.



Applying the Pythagorean Theorem, we obtain $DE = 10\sqrt{5}$ and $CF = 20\sqrt{3}$. Since $\triangle CTP$ is similar to $\triangle CDE$ and $\triangle DTP$ is similar to $\triangle DCF$, we have the equations

$$\frac{CT}{PT} = \frac{2}{\sqrt{5}} \quad \text{and} \quad \frac{TD}{PT} = \frac{1}{\sqrt{3}}.$$

Adding these equations and noting that $CT + TD = 20$ yields

$$\frac{20}{PT} = \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{3}},$$

from which we see that the unique solution is

$$PT = \frac{20\sqrt{15}}{\sqrt{5} + 2\sqrt{3}} = \frac{20(6\sqrt{5} - 5\sqrt{3})}{7}.$$

Also solved by Andrew Mao, A.B. Lucas Secondary School, London, ON; and Yifei Chen, West Windsor Plainsboro High School North, Plainsboro, NJ, USA.