MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier septembre 2004**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

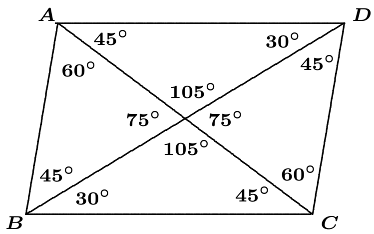
Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M132. *Proposé par Peter Y. Woo, Biola University, La Mirada, CA, USA.*

(a) Qui a-t-il de faux dans le dessin du parallélogramme ci-contre ? Les angles sont mesurés en degrés.

(b) Proposez une modification qui rendrait le dessin plausible. On ne doit pas modifier les segments issus de B.



M133. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Dans un pentagone $ABCDE$, chaque côté est parallèle à une diagonale. Montrer que le rapport d'une diagonale au côté parallèle correspondant est constant. En fait, cette constante est le nombre d'or. (Un tel pentagone est appelé *un pentagone d'or*.)

M134. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Dans un pentagone d'or $ABCDE$ (voir le problème précédent pour la définition), l'angle EAB est égal à l'angle BCD. Montrer que l'angle CDE est égal à l'angle DEA.

M135. *Proposé par l'Équipe de Mayhem.*

Trouver tous les nombres de deux chiffres avec exactement 8 diviseurs positifs.

M136. *Proposé par l'Équipe de Mayhem.*

Pour construire un nombre de cinq chiffres, on utilise chacun des chiffres 1, 4, 5, 7, et 8 une seule fois. Déterminer la somme de tous les différents nombres de cinq chiffres ainsi construits.

M137. *Proposé par Babis Stergiou, Lycio Psachnon Evias, Grèce.*

Soit a, b, $c > 0$, $a + b + c = 3$, et $abc = 1$.

(a) Montrer que $(a^2 + b)(a + b^2) \geq (a + a^2)(b + b^2)$;

(b) Déduire de (a) que (ou sinon montrer que)

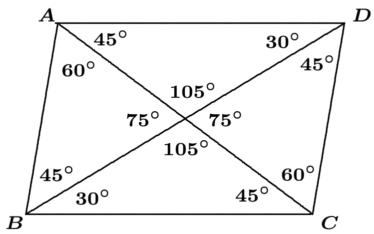
$$\frac{ab}{(a^2+b)(a+b^2)} + \frac{bc}{(b^2+c)(b+c^2)} + \frac{ca}{(c^2+a)(c+a^2)} \leq \frac{3}{4}.$$

. . . .

M132. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

(a) What is wrong with the diagram of the parallelogram? The angle measures are in degrees.

(b) Suggest a modification that would make the diagram plausible. You are not allowed to modify the three lines through B.



M133. *Proposed by K.R.S. Sastry, Bangalore, India.*

In pentagon $ABCDE$, each side is parallel to a diagonal. Show that the ratio of a diagonal to the corresponding parallel side is constant. In fact, this constant is the golden ratio. (Such a pentagon is called a *golden pentagon*.)

M134. *Proposed by K.R.S. Sastry, Bangalore, India.*

In golden pentagon $ABCDE$ (see the preceding problem for the definition), we have $\angle EAB = \angle BCD$. Show that $\angle CDE = \angle DEA$.

M135. *Proposed by the Mayhem staff.*

Find all two-digit numbers with exactly 8 positive divisors.

M136. *Proposed by the Mayhem staff.*

The digits 1, 4, 5, 7, and 8 are each used once to form a five-digit number. Determine the sum of all such distinct five-digit numbers.

M137. *Proposed by Babis Stergiou, Lycio Psachnon Evias, Greece.*

Suppose a, b, $c > 0$, $a + b + c = 3$, and $abc = 1$.

(a) Prove that $(a^2 + b)(a + b^2) \geq (a + a^2)(b + b^2)$.

(b) Hence, or otherwise, prove that

$$\frac{ab}{(a^2+b)(a+b^2)} + \frac{bc}{(b^2+c)(b+c^2)} + \frac{ca}{(c^2+a)(c+a^2)} \leq \frac{3}{4}.$$

Mayhem Solutions

M69. *Proposed by the Mayhem Staff.*

A sequence of digits is formed by writing the digits from the natural numbers in the order that they appear. The sequence starts:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, 1, 2, \ldots.$$

What is the 2002^{nd} digit in the sequence?

Solution by Alfian, grade 11 student, SMU Methodist, Palenbaug, Indonesia.

Writing the first 9 natural numbers requires 9 digits. Writing the next 90 numbers requires $90 \times 2 = 180$ digits. Thus, to write the numbers from 1 to 99 requires 189 digits. The next 9 groups of 100 natural numbers each require $100 \times 3 = 300$ digits. Clearly, only 6 of these groups are required. Once we have written the digits of the number 699, we will have used 1989 digits. Now, $2002 - 1989 = 13$ more digits are required. Therefore, the 2002^{nd} digit is the '7' from 704.

M70. *Proposed by the Mayhem Staff.*

What is the smallest positive multiple of 15 that is made up of only the digits 0, 4, and 7, each appearing the same number of times?

Solution by the Mayhem Staff.

Let n be the required positive integer. Clearly, $15|n$ implies both $5|n$ and $3|n$. Since $5|n$ and the only digits in n are 0, 4, and 7, it follows that n ends in the digit 0. Since $3|n$, the digital sum of n must be a multiple of 3, and thus, the number of repeats of each digit must be a multiple of 3. Since n is the *smallest* multiple of 15, we try the number of repeats equal to 3. Again, since we want the smallest positive integer, we need to have the smaller digits at the beginning of the integer. Thus, since leading zeroes are not allowed, $n = 400447770$.

M71. *Proposed by Richard Hoshino, Dalhousie University, Halifax, NS.*

Let $x = a + b - c$, $y = a + c - b$ and $z = b + c - a$, where a, b and c are prime numbers. Given that $x^2 = y$ and $\sqrt{z} - \sqrt{y}$ is the square of a prime number, determine all possible values for the product abc.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

The unique solution of the simultaneous linear equations $a+b-c = x$, $a + c - b = y$, and $b + c - a = z$ is

$$(a, b, c) = \left(\tfrac{1}{2}(x + y), \tfrac{1}{2}(x + z), \tfrac{1}{2}(y + z)\right) .$$

Setting $y = x^2$, we have

$$a = \tfrac{1}{2}(x + x^2)\,, \quad (1)$$
$$b = \tfrac{1}{2}(x + z)\,, \quad (2)$$
$$c = \tfrac{1}{2}(x^2 + z)\,. \quad (3)$$

From (1), we have $x = \dfrac{-1 \pm \sqrt{1 + 8a}}{2}$. Since x is an integer, we get $1 + 8a = T^2$, for some odd positive integer T. Hence, $2a = \dfrac{T - 1}{2} \cdot \dfrac{T + 1}{2}$. Therefore, since a is prime, we have $\dfrac{T - 1}{2} = 2$ and $\dfrac{T + 1}{2} = a$, from which we conclude that $T = 5$ and $a = 3$. Then, using (1), $x = 2$ or $x = -3$.

If $x = 2$, then $y = 4$ and $\sqrt{z} - 2 = p^2$, for some prime p. Thus, $z = \left(p^2 + 2\right)^2$. From (2), we see that z is even. Hence, $p = 2$ and $z = 36$. Substituting into (3), we get $c = 20$, which contradicts the fact that c is prime.

Therefore, $x = -3$. Then $y = 9$ and $z = \left(p^2 + 3\right)^2$, for some prime p. Now, (2) and (3) imply that z is odd. This happens only if $p = 2$, which gives $z = 49$. Then $b = 23$, and $c = 29$.

Then $abc = (3)(23)(29) = 2001$.

M72. *Proposed by J. Walter Lynch, Athens, GA, USA.*

You have a cup of coffee and a cup of tea. The cups are identical and each contains the same amount of liquid as the other. You take a teaspoon full of coffee out of the coffee cup and put it into the teacup. You then take a teaspoon full of the mixture out of the teacup and put it into the coffee cup. Which is greater, the amount of coffee in the teacup, the amount of tea in the coffee cup, or are they the same?

Solution by the Mayhem Staff.

Exactly 1 teaspoon of liquid was transferred in each direction; thus, the volume of liquid in each cup is the same both before and after the exchange. Suppose the amount of coffee in the teacup is greater than the amount of tea in the coffee cup. Then, since the total amount of coffee equals the total amount of tea, the remaining amount of coffee in the coffee cup is less than the remaining amount of tea in the teacup. This implies that the amount of liquid in the coffee cup is less than that in the teacup, which contradicts the statement in the first sentence.

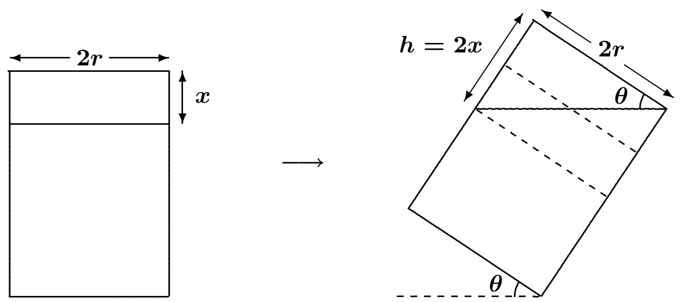
Similarly the supposition that the amount of coffee in the teacup is less than the amount of tea in the coffee cup leads to a contradiction.

Thus, these two amounts must be equal.

M73. *Proposed by J. Walter Lynch, Athens, GA, USA.*

A right circular cylinder with radius r and height h contains a liquid to within x of the top of the cylinder. Find the angle through which the cylinder must be tilted in order for the liquid to start to pour out. (Assume that there is enough liquid in the cylinder so that the surface of the liquid does not intersect the bottom of the cylinder before the liquid starts to pour out.)

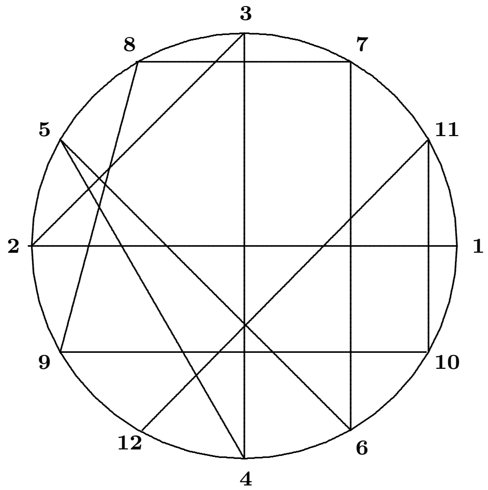
Solution by the Mayhem Staff.



As the cylinder is tilted, the volume of the unfilled space inside the cylinder remains constant as long as no liquid spills over. Consider the cylinder tilted to the point where liquid is just starting to spill (as in the figure above). Let θ denote the angle of tilt at this point. The liquid surface is now an ellipse. Draw a circle around the cylinder from the point on this ellipse which is diametrically opposite the lip of the cylinder. This circle marks off a cylinder of height h at the top of the original cylinder. The surface ellipse of the liquid divides this second cylinder into two congruent 'wedges', one of which is the unfilled space. Therefore, the volume of the entire second cylinder is twice that of the unfilled space. It follows that $h = 2x$. Hence, the angle θ is given by $\tan\theta = \frac{2x}{2r} = \frac{x}{r}$; that is, $\theta = \arctan\left(\frac{x}{r}\right)$.

M74. *Proposed by the Mayhem Staff.*

A circle has 12 equally spaced points placed on its circumference. How many ways can the numbers 1 through 12 be assigned to the points so that if the points 1 through 12 are connected with line segments, in order, the segments do not cross? An example of a **bad** arrangement is illustrated below.



Solution by the Mayhem Staff.

We will use the fact that there is at least one chord drawn to or from every point. We can place the label '1' at any of the 12 points. However, the label '2' is now forced to go at one of the two points adjacent to the point labelled '1', since, if we skip past an adjacent point, a chord to or from this adjacent point will intersect the chord joining the points that we have labelled '1' and '2'. Similarly, the label '3' can only be placed at one of the two points adjacent to the '1–2' block. There is a choice of two possible points for each successive label until the label '12', for which there is only one remaining point. Thus, there are $12\left(2^{10}\right) = 12\,288$ labellings.

M75. *Proposed by the Mayhem Staff.*

The increasing sequence 1, 5, 6, 25, 26, 30, 31, 125, 126, ... consists of positive integers that can be formed by adding distinct powers of 5. What is the 75^{th} integer in the sequence?

Solution by the Mayhem Staff.

We write the sequence in base 5 as 1, 10, 11, 100, 101, 110, ... If, instead of considering this sequence in base 5, we now consider it as a sequence of binary numbers, we note that the number in position n is n. Thus, since 75 may be written in binary as 1001011, the required number is $5^6 + 5^3 + 5 + 1 = 15\,756$.

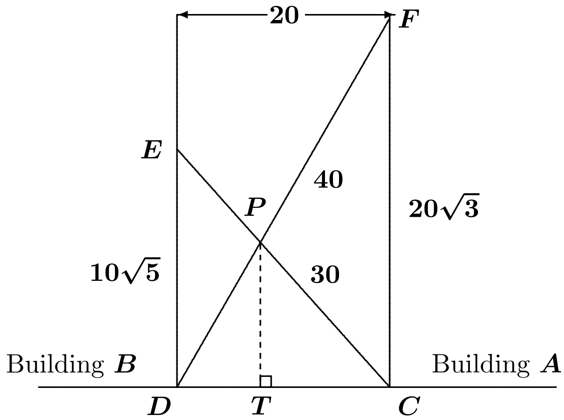
M76. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Two buildings A and B are twenty feet apart. A ladder thirty feet long has its lower end at the base of building A and its upper end against building B. Another ladder forty feet long has its lower end at the base of building B and its upper end against building A.

How high above the ground is the point where the ladders intersect?

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let CE and DF be the ladders, as shown in the diagram below. Let P be the point of intersection of the ladders, and let T be the point on the ground directly below P.



Applying the Pythagorean Theorem, we obtain $DE = 10\sqrt{5}$ and $CF = 20\sqrt{3}$. Since $\triangle CTP$ is similar to $\triangle CDE$ and $\triangle DTP$ is similar to $\triangle DCF$, we have the equations

$$\frac{CT}{PT} = \frac{2}{\sqrt{5}} \quad \text{and} \quad \frac{TD}{PT} = \frac{1}{\sqrt{3}} .$$

Adding these equations and noting that $CT + TD = 20$ yields

$$\frac{20}{PT} = \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{3}} ,$$

from which we see that the unique solution is

$$PT = \frac{20\sqrt{15}}{\sqrt{5} + 2\sqrt{3}} = \frac{20(6\sqrt{5} - 5\sqrt{3})}{7} .$$

Also solved by Andrew Mao, A.B. Lucas Secondary School, London, ON; and Yifei Chen, West Windsor Plainsboro High School North, Plainsboro, NJ, USA.

Pólya's Paragon

What's the difference?

Shawn Godin

"Little Johnnie encounters the following list of numbers 12, 49, 62, 57, 40, 17, ... *What is the next number in the list?"*

Questions like this have been seen in mathematics text books and on mathematics contests over the years. Is there a general way to attack them to find the next few terms?

It turns out that there is a powerful method called *finite differences* that deals with sequences quite well. The main idea is to find the differences between consecutive terms and look for a pattern. Many mathematical functions reveal their structure under this method. Let's see a couple of examples in action.

Example 1: Find the next term in the sequence 1, 3, 5, 7, 9, ...

Solution: This example is trivial, but it reveals our general technique. If we call the terms t_1, t_2, t_3, t_4, t_5, ... and the differences $d_1 = t_2 - t_1$, $d_2 = t_3 - t_2$, etc., we get the following table:

t_n	1		3		5		7		9
d_n		2		2		2		2	

In case we didn't see the pattern in the original sequence, the sequence of differences is "easier". We can produce the next term in the original sequence by realizing that $t_6 - t_5 = d_5 = 2$ (since all the differences are 2), and therefore,

$$t_6 = 2 + t_5 = 2 + 9 = 11 .$$

Technically, the differences that we produced are called *first differences*. If we rename our differences ${}_1d_1$, ${}_1d_2$, ${}_1d_3$, ${}_1d_4$, ${}_1d_5$, ..., then we can define the *second differences* as ${}_2d_1 = {}_1d_2 - {}_1d_1$, ${}_2d_2 = {}_1d_3 - {}_1d_2$, etc. These will be useful in the next problem.

Example 2: Find the next term in the sequence 1, 4, 9, 16, 25, ...

Solution: Again, we have a sequence that is easy to recognize. This time the first differences are not the same; hence, we will continue to the second differences.

t_n	1		4		9		16		25
${}_1d_n$		3		5		7		9	
${}_2d_n$			2		2		2		

If we didn't see a pattern in the first differences, we can easily see one in the second differences. The next second difference must be ${}_2d_4 = 2$, which gives the next first difference as ${}_1d_5 = 9 + 2 = 11$, and then the next term of the sequence is $t_6 = 25 + 11 = 36$.

At this time a pattern is emerging. Notice that when the sequence can be modelled by a linear function, the first differences are constant. Similarly, when the sequence can be modelled by a quadratic, the second differences are constant. This pattern continues:

Theorem. *If a sequence $\{t_n\}$ can be modelled by a polynomial of degree k, then the k^{th} differences are constant.*

We can see the proof of this by looking at a lemma first.

Lemma. *If a sequence $\{t_n\}$ can be modelled by a polynomial of degree k, then the first differences $\{{}_1d_n\}$ can be modelled by a polynomial of degree $k - 1$.*

Proof: We will only sketch the main idea (try to construct your own proof). Suppose we have a sequence $\{t_n\}$ where $t_n = n^k$. Then

$$ {}_1d_n \;=\; t_{n+1} - t_n \;=\; (n+1)^k - n^k \,. $$

Using the Binomial Theorem, we can see that ${}_1d_n$ is of degree $k - 1$.

Using the lemma, you can prove the theorem by induction. With a little experimentation, you will also see that the difference that is constant can be used to determine the coefficient of the highest degree term. See if you can discover the connection, and provide a proof. With this new theorem, you should be set to attack the original sequence.

We will return to this topic next issue and expand on it. Here are some problems to keep you busy in the meantime (homework, one might say).

Examine the differences for the following sequences, and see if you can make any predictions.

1. $a_n = 2^n$.
2. $b_n = 5^n$.
3. $c_n = 2 \times 3^n$.
4. $1, 1, 2, 3, 5, 8, 13, \ldots$. (This is the *Fibonacci sequence*. Each term is the sum of the previous two terms.)

It should be noted, that any sequence like the one with which we started can be continued in **any** *way. That is, we can pick any number to go next and find a polynomial that will match those numbers. See the article on Lagrange Interpolation from the Skoliad Corner* [2001 : 386–388].

The Birthday Problem Revisited

Sandra M. Pulver

Problems about birthday probabilities are among the most interesting problems in probability due to their surprising answers, and they are more germane to the beginning probability student than many of the other problems in elementary probability texts. We will discuss three of the many birthday problems in existence.

In these problems, February 29$^{\text{th}}$ is ignored as a possible birthday and the other 365 days are treated as equally likely.

The Classical Birthday Problem. *What is the minimum number of people so that the probability of two or more of them having the same birthday exceeds one half?*

We need to find the probability that, among n people, at least two have a common birthday. We will solve the complementary problem of finding the probability that *no two people have the same birthday*.

Since each person has 365 possible birthdays, the number of possible ways for n people to have birthdays is 365^n. The number of ways for n people to have no matching birthdays is

$$365 \times 364 \times \cdots \times (365 - n + 1)\,,$$

because there are 365 possible birthdays for the first person, 364 possible *different* birthdays for the second person, 363 for the third person, and so on. The last person can have a birthday different from the first $(n-1)$ people in $365 - (n - 1)$ ways. Thus, the probability that no two of the n people have the same birthday is

$$\frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}\,.$$

Now the probability P_n of at least one matching pair of birthdays among the n people is

$$P_n \;=\; 1 - \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}\,.$$

According to our objective, we have to find the smallest n such that $P_n > \frac{1}{2}$. The following table shows that the answer is 23.

n	5	10	15	20	21	22	23	24	25
P_n	0.027	0.117	0.253	0.411	0.444	0.476	0.507	0.538	0.569

The Birthmate Problem. *What is the smallest number of strangers you need to meet in order to have at least a 50% chance of finding one whose birthday is the same as yours?*

This time, we need to find the probability that, among n strangers, at least one has the same birthday as yours. We again consider the complementary problem of finding the probability that no birthdays are the same as yours.

First of all, the probability that some birthday is different from yours is $\frac{364}{365}$, or $\frac{N-1}{N}$, if we set $N = 365$. Since we have n strangers, the probability that nobody among them has your birthday is $\left(\frac{N-1}{N}\right)^n$. Then the probability P_n that at least one person's birthday matches yours is

$$P_n = 1 - \left(\frac{N-1}{N}\right)^n = 1 - \left(1 - \frac{1}{N}\right)^n .$$

As N increases, the value of $\left(1 - \frac{1}{N}\right)^N$ approaches e^{-1}. We therefore have the approximation

$$\left(1 - \frac{1}{N}\right)^N \approx e^{-1} .$$

Taking the N^{th} root of both sides, we obtain $1 - \frac{1}{N} \approx e^{-\frac{1}{N}}$. Now,

$$\left(1 - \frac{1}{N}\right)^n \approx \left(e^{-\frac{1}{N}}\right)^n = e^{-\frac{n}{N}} .$$

Hence, $P_n \approx 1 - e^{-\frac{n}{N}}$.

We want $P_n \geq \frac{1}{2}$, which implies $e^{-\frac{n}{N}} \leq \frac{1}{2}$. Taking natural logarithms on both sides, we have

$$-\frac{n}{N} \approx \ln\left(\tfrac{1}{2}\right) \approx -0.693 ,$$

or $n \approx 0.693N$. Since $N = 365$, we have

$$n \approx 252.945 .$$

Since n must be an integer, it must be 253. Thus, we need to ask at least 253 strangers to have at least a 50% chance that one of their birthdays will coincide with our own.

Birthmate Problem for a Group. *If n people meet by chance, what is the probability that they all have the same birthday?*

Observe that the probability that any given day is the birthday for a given person is $\frac{1}{365}$. Therefore, the probability that n persons all have a particular birthday is $\frac{1}{365^n}$. Now, adding over all 365 possible birthdays, the probability that all n persons have the same birthday is

$$365\left(\frac{1}{365^n}\right) = \frac{1}{365^{n-1}}.$$

To this point we have assumed that we were ignoring leap years. This allowed us to simplify the calculations. Readers may wish to reconsider the previous examples without the assumption. In this final example, if we were working with a leap year instead, the probability would change to $\frac{1}{366^{n-1}}$. In order to consider both possibilities, we look at a 4–year cycle, in which case the probability can be seen to be $(1 + 365 \times 4^n)/1461^n$. All three of these probabilities are computed in the following table for selected values of n.

n	2	10	20
non-leap year	0.002740	8.697×10^{-24}	2.072×10^{-49}
leap year	0.002732	8.485×10^{-24}	1.967×10^{-49}
overall	0.002736	8.638×10^{-24}	2.044×10^{-49}

Therefore, for all n people to have the same birthday is very unlikely.

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