THE OLYMPIAD CORNER

No. 236

R.E. Woodrow

As a first set of Olympiad problems, we give the Hungary-Israel Binational Mathematical Competition 2001. Thanks go to Chris Small, Canadian Team Leader to the 46^{th} IMO, for collecting them.

HUNGARY-ISRAEL BINATIONAL MATHEMATICAL COMPETITION 2001

Individual Competiton

1. Find positive integers x, y, z such that $x > z > 1999 \cdot 2000 \cdot 2001 > y$ and $2000x^2 + y^2 = 2001z^2$.

2. Points A, B, C, D lie on the line ℓ, in that order. Find the locus of points P in the plane for which $\angle APB = \angle CPD$.

3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real x,

$$f(f(x)) = f(x) + x \,.$$

4. Let $P(x) = x^3 - 3x + 1$. Find the polynomial Q whose roots are the fifth power of the roots of P.

5. A triangle ABC is given. The mid-points of sides AC and AB are B_1 and C_1, respectively. The centre of the incircle of $\triangle ABC$ is I. The lines B_1I, B_2I meet the sides AB, AC at C_2, B_2, respectively. Given that the areas of $\triangle ABC$ and $\triangle AB_2C_2$ are equal, what is $\angle BAC$?

6. Given are 32 positive integers with a sum of 120, one of which is greater than 60. Prove that these integers can be divided into two disjoint subsets that have the same sum.

Team Competiton

In the following questions, G_n is a simple undirected graph with n vertices, K_n is the complete graph with n vertices, $K_{n,m}$ is the complete bipartite graph with m vertices in one of the two partite sets and n vertices in the other, and C_n is a circuit with n vertices. The number of edges in the graph G_n is denoted $e(G_n)$.

1. The edges of K_n, $n \geq 3$, are coloured with n colours, and every colour appears at least once. Prove that there is a triangle whose sides are coloured with 3 different colours.

2. An integer $n \geq 5$ is given. If $e(G_n) \geq \frac{n^2}{4} + 2$, prove that there exist two triangles which have exactly one common vertex.

3. If $e(G_n) \geq \frac{n\sqrt{n}}{2} + \frac{n}{4}$, prove that G_n contains C_4.

4. (a) If G_n does not contain $K_{2,3}$, prove that $e(G_n) \leq \frac{n\sqrt{n}}{\sqrt{2}} + n$.

(b) Given $n \geq 16$ distinct points P_1, P_2, $\dots$, P_n in the plane, prove that at most $n\sqrt{n}$ of the segments P_iP_j have unit length.

5. (a) Let p be a prime. Consider the graph whose vertices are the ordered pairs (x, y) with x, $y \in \{0, 1, 2, \dots, p-1\}$, and whose edges join vertices (x, y) and (x', y') if and only if $xx' + yy' \equiv 1 \pmod{p}$. Prove that this graph does not contain C_4.

(b) Prove that for infinitely many values of n, there is a graph G_n that does not contain C_4 and satisfies $e(G_n) \geq \frac{n\sqrt{n}}{2} - n$.

Next we turn to the problems of the Second Hong Kong (China) Mathematical Olympiad written December 1999. Thanks again go to Chris Small, Canadian Team Leader to the 46th IMO, for forwarding them to us.

SECOND HONG KONG (CHINA) MATHEMATICAL OLYMPIAD

December 1999

Time: 3 hours

1. [5 marks] Determine all positive rational numbers $r \neq 1$ such that $r^{1/(r-1)}$ is rational.

2. [10 marks] Let I and O be the incentre and circumcentre, respectively, of $\triangle ABC$. Assume $\triangle ABC$ is not equilateral (so that $I \neq O$). Prove that $\angle AIO \leq 90^\circ$ if and only if $2BC \leq AB + CA$.

3. [10 marks] Students have taken a test in each of n subjects ($n \geq 3$). It is known that, for any subject, exactly three students got the best score in the subject, and for any two subjects, exactly one student got the best score in both of the subjects. Determine the smallest n so that the above conditions imply that exactly one student got the best score in all n subjects.

4. [10 marks] Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x, y \in \mathbb{R}$,

$$f(x + yf(x)) \;=\; f(x) + xf(y)\,.$$

A third set of problems for your puzzling pleasure are those of the 17th Balkan Mathematical Olympiad, written in May, 2000. Thanks again go to Chris Small.

17th BALKAN MATHEMATICAL OLYMPIAD

Chisinau, Republic of Moldova

May 5, 2000 — Time: 4.5 hours

Each problem is worth 10 points.

1. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f(xf(x) + f(y)) \;=\; \left(f(x)\right)^2 + y\,,$$

for any real numbers x and y.

2. Let ABC be a non-isosceles acute triangle, and let E be an interior point of the median AD, with D on BC. Let F be the orthogonal projection of E onto the line BC. Let M be an interior point of the segment EF, and let N and P be the orthogonal projections of M onto the lines AC and AB, respectively. Prove that the bisectors of angles PMN and PEN are parallel.

3. Find the maximal number of rectangles of size $1 \times 10\sqrt{2}$ which can be cut off from a rectangle of size 50×90 using cuts parallel to the edges of the initial 50×90 rectangle.

4. We say that a positive integer r is a *power* if it has the form $r = t^s$, for some integers $t \geq 2$ and $s \geq 2$. Show that, for any positive integer n, there exists a set A of n positive integers which satisfies the following conditions:

(i) Every element of A is a power.

(ii) For any k elements $r_1, r_2, \ldots, r_k$ from A (where $2 \leq k \leq n$), the number $\dfrac{r_1 + r_2 + \cdots + r_k}{k}$ is a power.

Now we turn to our readers for solutions to problems of the Hungary-Israel Mathematical Competition 1999 given [2001: 421–422].

1. Let $f(x)$ be a polynomial whose degree is at least 2. Define the sequence $g_i(x)$ by: $g_1(x) = f(x)$ and $g_{n+1}(x) = f(g_n(x))$ for $n = 1, 2, \ldots$. Let r_n be the average of the roots of $g_n(x)$. It is given that $r_{19} = 99$. Find r_{99}.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution, modified slightly by the editor.

Let $f(x) = \sum\limits_{i=0}^{p} c_i x^i$, where $p \geq 2$. Since $g_n = \underbrace{f \circ f \circ \ldots \circ f}_{n}$, we easily deduce that g_n is a polynomial of degree p^n. In $g_n(x)$, let α_n and β_n be the coefficients of x^{p^n} and x^{p^n-1}, respectively. Since g_n has p^n roots,

$$r_n = -\frac{\beta_n}{p^n \alpha_n}. \tag{1}$$

For each $n \geq 1$,

$$g_{n+1}(x) = f\big(g_n(x)\big) = \sum_{i=0}^{p} c_i\big(g_n(x)\big)^i = c_p\big(g_n(x)\big)^p + \sum_{i=0}^{p-1} c_i\big(g_n(x)\big)^i.$$

The degree of $\sum\limits_{i=0}^{p-1} c_i\big(g_n(x)\big)^i$ is at most $(p-1)p^n$. Therefore, terms of orders p^{n+1} and $p^{n+1}-1$ appear only in the expansion of $c_p\big(g_n(x)\big)^p$. We have

$$c_p\big(g_n(x)\big)^p = c_p\left(\alpha_n x^{p^n} + \beta_n x^{p^n-1} + h_n(x)\right)^p,$$

where $\deg(h_n) \leq p^n - 2$. Applying the Binomial Theorem, we deduce that

$$\alpha_{n+1} = c_p \alpha_n^p \quad \text{and} \quad \beta_{n+1} = p c_p \alpha_n^{p-1} \beta_n.$$

Then, using (1), we have

$$r_{n+1} = -\frac{\beta_{n+1}}{p^{n+1}\alpha_{n+1}} = -\frac{p c_p \alpha_n^{p-1}\beta_n}{p^{n+1} c_p \alpha_n^p} = -\frac{\beta_n}{p^n \alpha_n} = r_n.$$

By induction, $r_n = r_1$ for all $n \in \mathbb{N}$. Thus, $r_{99} = r_{19} = 99$.

2. A set of $2n+1$ lines in a plane is drawn. No two of them are parallel, and no three pass through one point. Every three of these lines form a non-right triangle. Determine the maximal number of acute-angled triangles that can be formed.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let ℓ_1, ℓ_2, $\dots$, ℓ_{2n+1} be the lines, and let M_{ij} be the common point of ℓ_i and ℓ_j (for $i \neq j$). Consider that we are in the complex plane $(O, \overrightarrow{u}, \overrightarrow{v})$.

Let L be an arbitrary line, distinct from and not parallel to any of the ℓ_i's, and such that none of the M_{ij}'s belongs to L. Let α_i be the angle between the lines L and $\ell_i \pmod{\pi}$.

Then, the triangle formed by the lines ℓ_i, ℓ_j, ℓ_k is $M_{ij}M_{jk}M_{ki}$, and

$$2(\overrightarrow{M_{ij}M_{ik}}, \overrightarrow{M_{ij}M_{kj}}) \;=\; 2(L,\ell_i) + 2(\ell_j, L) \;=\; 2(\alpha_i - \alpha_j) \pmod{2\pi}\ .$$

Let P_i be the point on the unit circle Γ with center O such that

$$(\overrightarrow{u}, \overrightarrow{OP_i}) \;=\; 2\alpha_i \pmod{2\pi}\ .$$

Then, for i, j, k pairwise distinct, we have

$$\begin{aligned} 2(\overrightarrow{P_kP_j}, \overrightarrow{P_kP_i}) &= (\overrightarrow{OP_j}, \overrightarrow{OP_i}) \;=\; 2(\alpha_i - \alpha_j) \\ &= 2(\overrightarrow{M_{ij}M_{ik}}, \overrightarrow{M_{ij}M_{kj}}) \pmod{2\pi}\ . \end{aligned}$$

Thus, $\angle P_iP_kP_j = \angle M_{ik}M_{ij}M_{kj}$. It follows that $\angle M_{ik}M_{ij}M_{kj}$ is acute if and only if $\angle P_iP_kP_j$ is acute, and $\angle M_{ik}M_{ij}M_{kj}$ is obtuse if and only if $\angle P_iP_kP_j$ is obtuse (since there is no right triangle).

Moreover, we note that if points P_1, P_2, $\dots$, P_{2n+1} are given on Γ, then we may find some lines ℓ_1, ℓ_2, $\dots$, ℓ_{2n+1} such that the construction above leads to the given P_i's.

Therefore, the problem is equivalent to finding the maximum number of acute triangles formed by the P_i's. That is, it is equivalent to finding the minimum number of obtuse triangles formed by these points. But the triangle $P_iP_kP_j$ is obtuse if and only if P_i, P_j, and P_k belong to one semicircle defined on Γ.

Let $i \in \{1, 2, \dots, 2n+1\}$ be fixed. Suppose that there are d_i points on one side of the diameter with endpoint P_i. Then there are $2n - d_i$ points on the other side of it (since there is no right triangle). The number of obtuse triangles with vertex P_i and an acute angle at P_i is

$$\begin{aligned} N_i &= \binom{d_i}{2} + \binom{2n-d_i}{2} \;=\; d_i^2 + 2n^2 - 2nd_i - n \\ &= (d_i - n)^2 + n^2 - n \;\geq\; n^2 - n\,, \end{aligned}$$

with equality if and only if $d_i = n$.

Summing over i, we count each obtuse triangle exactly twice. Thus, the number of obtuse triangles is

$$N \;=\; \frac{1}{2}\sum_{i=1}^{2n+1} N_i \;\geq\; \frac{(2n+1)n(n-1)}{2}\,.$$

Since there are exactly $\binom{2n+1}{3} = \frac{(2n+1)n(2n-1)}{3}$ triangles formed by the P_i's, it follows that the number of acute triangles is at most

$$\frac{(2n+1)n(2n-1)}{3} - \frac{(2n+1)n(n-1)}{2} = \frac{n(n+1)(2n+1)}{6}.$$

This value is achieved if, for example, we choose the P_i's as the vertices of a regular $(2n+1)$–gon inscribed in Γ.

Thus, the maximal number of acute triangles is $\frac{n(n+1)(2n+1)}{6}$.

3. Find all the functions f from the set of rational numbers to the set of real numbers such that for all rational x, y,

$$f(x+y) = f(x)f(y) - f(xy) + 1.$$

Solved by Michel Bataille, Rouen, France; and Christopher J. Bradley, Bristol, UK. We give Bataille's write-up.

The functions $x \mapsto 1$ and $x \mapsto x+1$ are clearly solutions. We now show that there is no other solution.

Suppose $f : \mathbb{Q} \to \mathbb{R}$ satisfies

$$f(x+y) = f(x)f(y) - f(xy) + 1 \qquad (1)$$

for all x, $y \in \mathbb{Q}$. Taking $x = y = 0$, we get $f(0) = 1$. Then, taking $y = -x$ (for any $x \in \mathbb{Q}$), we get

$$f(x)f(-x) = f(-x^2). \qquad (2)$$

From (2), $f(-1) = 0$ or $f(1) = 1$.

If $f(1) = 1$, then, using (1),

$$f(x) = f\big((x-1)+1\big) = f(x-1)f(1) - f(x-1) + 1 = 1,$$

for all $x \in \mathbb{Q}$. Hence, f is the constant function $x \mapsto 1$.

Suppose now that $f(-1) = 0$, and let $a = f(1)$. Taking $x = y = -1$ in (1) gives $f(-2) = \big(f(-1)\big)^2 - f(1) + 1 = 1 - a$. Then, taking $x = 1$ and $y = -2$ in (1), we get

$$\begin{aligned} f(-1) &= f(1)f(-2) - f(-2) + 1, \\ 0 &= a(1-a) - (1-a) + 1, \\ &= a(2-a). \end{aligned}$$

Therefore, $a = 0$ or $a = 2$.

If $a = 0$ (that is, $f(1) = 0$), then, from (1),

$$\begin{aligned} f(x) &= f(x-1)f(1) - f(x-1) + 1 = 1 - f(x-1) \\ &= 1 - \big(f(x)f(-1) - f(-x) + 1\big) = f(-x), \end{aligned}$$

showing that f is even. Then (2) gives $\left(f(\frac{1}{2})\right)^2 = f\left(\frac{1}{4}\right)$. It follows that

$$a \;=\; f\left(\tfrac{1}{2}+\tfrac{1}{2}\right) \;=\; \left(f\left(\tfrac{1}{2}\right)\right)^2 - f\left(\tfrac{1}{4}\right) + 1 \;=\; 1\,,$$

contradicting $a = 0$.

Thus, $f(1) = 2$. Using (1), we deduce that $f(x+1) = f(x) + 1$. By an easy induction, $f(x+n) = f(x) + n$ for all $x \in \mathbb{Q}$ and $n \in \mathbb{N}$. Recalling that $f(0) = 1$, we get $f(n) = n + 1$. Then, using (1) again,

$$\begin{aligned} f(x+n) &= f(x)f(n) - f(nx) + 1\,, \\ f(x) + n &= (n+1)f(x) - f(nx) + 1\,, \\ f(nx) &= nf(x) - n + 1\,. \end{aligned}$$

Now, let $r = m/n$, where m, $n \in \mathbb{N}$. Then

$$m + 1 \;=\; f(m) \;=\; f(nr) \;=\; nf(r) - n + 1\,.$$

Thus, $f(r) = \dfrac{m+n}{n} = 1 + r$. Moreover,

$$\begin{aligned} f(-r) &= f\big((1-r) + (-1)\big) \;=\; f(1-r)f(-1) - f(r-1) + 1 \\ &= -\big(f(r) - 1\big) + 1 \;=\; 2 - f(r) \;=\; 1 + (-r)\,. \end{aligned}$$

As a result, $f(x) = x + 1$ for all $x \in \mathbb{Q}$, and the proof is complete.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is almost equivalent to problem #3 of the Mathematical Olympiad in Bosnia and Herzegovina 1997 [2000 : 326]. From it, one can show that the two solutions are $f(x) = 1$ and $f(x) = x + 1$.

4. Let c be a positive integer. Define the following sequence:

$$a_1 \;=\; c, \quad a_{n+1} \;=\; ca_n + \sqrt{(c^2-1)(a_n^2-1)}\,, \quad n = 1, 2, \ldots.$$

Prove that all the terms a_n are positive integers.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bataille's approach.

Since $c \geq 1$, we have $a_1 = c = \cosh(x)$ for some non-negative real number x. Induction now shows that $a_n = \cosh(nx)$ for all n. Indeed, if $a_n = \cosh(nx)$ for some n, then

$$\begin{aligned} a_{n+1} &= \cosh(x)\cdot\cosh(nx) + \sqrt{\big(\sinh(x)\big)^2\big(\sinh(nx)\big)^2} \\ &= \cosh(x)\cdot\cosh(nx) + \sinh(x)\cdot\sinh(nx) \\ &= \cosh\big((n+1)x\big)\,. \end{aligned}$$

It follows that $a_{n+2} + a_n = 2\cosh(x) \cdot \cosh\big((n+1)x\big) = 2ca_{n+1}$. Now, taking into account that $a_1 = c \in \mathbb{Z}$ and $a_2 = 2c^2 - 1 \in \mathbb{Z}$, and using the relation $a_{n+2} = 2ca_{n+1} - a_n$, an immediate induction shows that $a_n \in \mathbb{Z}$ for all n. The result follows, since $a_n = \cosh(nx) \geq 1$.

5. The function

$$f(x,y,z) = \frac{x^2+y^2+z^2}{x+y+z}$$

is defined for every x, y, z such that $x+y+z \neq 0$. Find a point (x_0, y_0, z_0) such that $0 < x_0^2 + y_0^2 + z_0^2 < \frac{1}{1999}$ and $1.999 < f(x_0, y_0, z_0) < 2$.

Solution by Christopher J. Bradley, Bristol, UK.

A solution is

$$(x_0, y_0, z_0) = (0.0009998, -0.0009998, 0.000001).$$

Note that $x_0 + y_0 + z_0 = 0.000001$ and $x_0^2 + y_0^2 + z_0^2 = 0.0000019992\ldots$ Thus, $f(x_0, y_0, z_0) = 1.9992\ldots$, which lies between 1.999 and 2.

(This solution required only one "difficult" calculation, namely $(0.0009998)^2 = (0.001)^2(1-0.0002)^2$ —which is not all that difficult.)

Now we turn to solutions to problems of the 12th Korean Mathematical Olympiad, Final Round, given [2001 : 422–423].

1. Let R, r be the circumradius, and the inradius of $\triangle ABC$, respectively, and let R', r' be the circumradius and inradius of $\triangle A'B'C'$, respectively. Prove that if $\angle C = \angle C'$ and $Rr' = R'r$, then the two triangles are similar.

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution.

It is well known (see [2001 : 46]) that

$$\cos A + \cos B + \cos C = 1 + \frac{R}{r}.$$

Since $\dfrac{R}{r} = \dfrac{R'}{r'}$, we have

$$\cos A + \cos B + \cos C = \cos A' + \cos B' + \cos C'.$$

Since $C = C'$, we deduce that $\cos A + \cos B = \cos A' + \cos B'$; that is,

$$2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2\cos\left(\frac{A'+B'}{2}\right)\cos\left(\frac{A'-B'}{2}\right).$$

Now, note that $A + B = \pi - C = \pi - C' = A' + B'$. Thus,

$$\cos\left(\frac{A-B}{2}\right) \;=\; \cos\left(\frac{A'-B'}{2}\right) .$$

With no loss of generality, we may suppose that $A \geq B$ and $A' \geq B'$. Then $A - B = A' - B'$. Since we also have $A + B = A' + B'$, we deduce that $A = A'$ and $B = B'$. Then $\triangle ABC$ is similar to $\triangle A'B'C'$.

2. Suppose $f(x)$ is a function satisfying $|f(m+n) - f(m)| \;\leq\; \frac{n}{m}$ for all rational numbers n and m. Show that for all natural numbers k

$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \;\leq\; \frac{k(k-1)}{2} .$$

Solved by Mohammed Aassila, Strasbourg, France; and Christopher J. Bradley, Bristol, UK. We give Aassila's write-up.

For all natural numbers i, we have

$$\left|f(2^{i+1}) - f(2^i)\right| \;=\; \left|f(2^i + 2^i) - f(2^i)\right| \;\leq\; \frac{2^i}{2^i} \;=\; 1 .$$

For any integer $k > i$,

$$f(2^k) - f(2^i) \;=\; \sum_{j=i}^{k-1} \left(f(2^{j+1} - f(2^j)\right) ,$$

and hence,

$$\left|f(2^k) - f(2^i)\right| \;\leq\; \sum_{j=i}^{k-1} \left|f(2^{j+1}) - f(2^j)\right| \;\leq\; \sum_{j=i}^{k-1} 1 \;=\; k - i .$$

Consequently,

$$\sum_{i=1}^{k} \left|f(2^k) - f(2^i)\right| \;=\; \sum_{i=1}^{k-1} \left|f(2^k) - f(2^i)\right| \;\leq\; \sum_{i=1}^{k-1} (k-i) \;=\; \frac{k(k-1)}{2} .$$

3. Find all positive integers n such that $2^n - 1$ is a multiple of 3 and $\frac{2^n-1}{3}$ is a divisor of $4m^2 + 1$ for some integer m.

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bradley's solution.

We will prove that the positive integers n which satisfy the given conditions are those of the form 2^t, for positive integers t. The following lemmas will be needed. For proofs of Lemmas 2 and 3, we refer to Niven, Zuckerman and Montgomery, *An Introduction to the Theory of Numbers* (Wiley, 1991).

Lemma 1. For any positive integer n, the positive integer $2^n - 1$ is divisible by 3 if and only if n is even.

Proof. If $n = 2m$ for some positive integer m, then $2^n = 2^{2m} = 4^m$. Thus, $2^n - 1 = 4^m - 1$ which is divisible by $4 - 1 = 3$.

If $n = 2m - 1$ for some positive integer m, then

$$2^n - 1 \;=\; 2^{2m-1} - 1 \;=\; 1 + 2 + 2^2 + \cdots + 2^{2m-2} \;\equiv\; 1 \pmod 3 ,$$

since $2^{2k-1} + 2^{2k} = 3(2^{2k-1}) \equiv 0 \pmod 3$ for all positive integers k.

Lemma 2. If q is a prime of the form $4k + 3$ and $q \mid a^2 + b^2$, then $q \mid a$ and $q \mid b$.

Lemma 3. If $N = a^2 + b^2$ is odd and contains no prime factors of the form $4k + 3$, then there exist *coprime* integers x, y such that $N = x^2 + y^2$. (In fact, the number of such representations is 2^{t+2}, where t is the number of primes p of the form $4k + 1$ that divide N.)

In view of Lemma 1, our problem reduces to finding all *even* positive integers n such that $\frac{1}{3}(2^n - 1)$ divides $4m^2 + 1$ for some integer m. Let $n = 2k$, where k is a positive integer. Then $\frac{1}{3}(2^n - 1) = \frac{1}{3}(4^k - 1)$. We will show that $\frac{1}{3}(4^k - 1)$ divides $4m^2 + 1$ for some integer m if and only if $k = 2^t$ for some integer t.

Let k be a positive integer, and let $N = \frac{1}{3}(4^k - 1)$.

First we suppose that $k = 2^t$ for some (non-negative) integer t. Then

$$\begin{aligned} N &= \tfrac{1}{3}(4^{2^t} - 1) \;=\; 1 + 4 + 4^2 + \cdots + 4^{(2^t-1)} \\ &= (1 + 4)(1 + 4^2)(1 + 4^4)\cdots(1 + 4^{2^{t-1}}) , \end{aligned}$$

since $1 + 2 + 4 + \cdots + 2^{t-1} = 2^t - 1$. In this factorization of N, each factor $1 + 4^u = 1^2 + (2^u)^2$ is odd and is a sum of two squares. Therefore, N is odd and, by a well-known theorem, is expressible as a sum of two squares, one even and one odd. By Lemma 2, the factors $1 + 4^u$ contain no prime factor of the form $4k + 3$, since such a factor cannot divide 1. Therefore, N contains no prime factor of the form $4k+3$. By Lemma 3, it follows that $N = x^2 + y^2$ for some x, y such that $(x, y) = 1$. Furthermore, one of x, y is odd and the other is even.

Since $(x, y) = 1$, there exist integers λ and μ such that $\lambda x + \mu y = 1$. For any such λ and μ,

$$\begin{aligned} N(\lambda^2 + \mu^2) &= (x^2 + y^2)(\lambda^2 + \mu^2) \\ &= (\lambda x + \mu y)^2 + (\lambda y - \mu x)^2 \\ &= 1 + (\lambda y - \mu x)^2 . \end{aligned}$$

We will show that λ and μ may be chosen so that $\lambda y - \mu x$ is even. Then we will have $N(\lambda^2 + \mu^2) = 1 + 4m^2$ for an integer m, showing that N is a divisor of $1 + 4m^2$.

If λ_0 and μ_0 are particular values of λ and μ such that $\lambda_0 x + \mu_0 y = 1$, then the general λ and μ are given by $\lambda = \lambda_0 - ky$ and $\mu = \mu_0 + kx$, where k is any integer. Then $\lambda y - \mu x = \lambda_0 y - \mu_0 x - k(x^2 + y^2)$. Therefore, if $\lambda_0 y - \mu_0 x$ is odd, we can choose $k = 1$, making $\lambda y - \mu x$ even.

Now suppose k is not a power of 2. Then $k = 2^s t$, where t is odd and $t \geq 3$, and we have

$$N \;=\; \tfrac{1}{3}(4^{2^s t} - 1) \;=\; \tfrac{1}{3}(2^{2^{s+1}t} + 1)(2^{2^{s+1}t} - 1)\,.$$

The second factor is divisible by $2^t - 1$, which is congruent to $3 \pmod 4$ and to $1 \pmod 3$. In other words, this factor contains a prime factor p other than 3 such that $p \equiv 3 \pmod 4$. Hence, N contains this factor p. If N has the form x^2+y^2, then x and y both have the factor p, by Lemma 2. Therefore, there is no multiple of N that is equal to $1 + 4m^2$ for some integer m (since p cannot divide 1).

4. Suppose that for any real x ($|x| \neq 1$), a function $f(x)$ satisfies

$$f\left(\frac{x-3}{x+1}\right) + f\left(\frac{3+x}{1-x}\right) \;=\; x\,.$$

Find all possible $f(x)$.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Yuming Chen and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use the solution of Chen and Wang.

Direct computation shows that the given condition is satisfied by

$$f(x) \;=\; \frac{7x + x^3}{2(1-x^2)}\,, \qquad x \;\neq\; \pm 1\,.$$

We will prove that this is the only possible $f(x)$.

Let $f(x)$ be any function that satisfies the given condition. Let $x \in \mathbb{R}$ be arbitrary such that $x \neq \pm 1$. Set $y = \dfrac{x+3}{1-x}$. Then it is readily seen that $y \neq \pm 1$ and $x = \dfrac{y-3}{y+1}$. Furthermore, we find that $\dfrac{3+y}{1-y} = \dfrac{x-3}{x+1}$. Hence,

$$f(x) + f\left(\frac{x-3}{x+1}\right) \;=\; f\left(\frac{y-3}{y+1}\right) + f\left(\frac{3+y}{1-y}\right) \;=\; y\,;$$

that is,

$$f(x) + f\left(\frac{x-3}{x+1}\right) \;=\; \frac{x+3}{1-x}\,. \tag{1}$$

Now set $y = \frac{x-3}{x+1}$. It is readily seen that $y \neq \pm 1$ and $x = \frac{3+y}{1-y}$. Furthermore, $\frac{y-3}{y+1} = \frac{3+x}{1-x}$. Hence,

$$f\left(\frac{3+x}{1-x}\right) + f(x) \;=\; f\left(\frac{y-3}{y+1}\right) + f\left(\frac{3+y}{1-y}\right) \;=\; y\,;$$

that is,

$$f\left(\frac{3+x}{1-x}\right) + f(x) \;=\; \frac{x-3}{x+1}\,. \qquad (2)$$

Adding (1) and (2), we then have

$$2f(x) + x \;=\; \frac{x+3}{1-x} + \frac{x-3}{x+1} \;=\; \frac{8x}{1-x^2}\,,$$

from which it follows that

$$f(x) \;=\; \frac{1}{2}\left(\frac{8x}{1-x^2} - x\right) \;=\; \frac{7x+x^3}{2(1-x^2)}\,.$$

5. Consider a permutation $a_1a_2a_3a_4a_5a_6$ of the 6 numbers $\{1, 2, 3, 4, 5, 6\}$ which can be transformed to $1\;2\;3\;4\;5\;6$ by transposing two numbers exactly 4 times (not less than 4 times). Find the number of such permutations.

Solution by Christopher J. Bradley, Bristol, UK.

In the theory of permutations, every permutation may be expressed as a product of cycles. For example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 2 & 6 & 4 \end{pmatrix} \;=\; (1\;5\;6\;4\;2\;3)\,,$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 1 & 3 & 2 \end{pmatrix} \;=\; (2\;6)(1\;5\;3\;4)\,.$$

There are $(n-1)!$ different n–cycles. For example, the number of different cycles of $\{1, 5, 3, 4\}$ is $3! = 6$, namely $(1\;5\;3\;4)$, $(1\;5\;4\;3)$, $(1\;4\;5\;3)$, $(1\;4\;3\;5)$, $(1\;3\;4\;5)$, and $(1\;3\;5\;4)$. (Any one of the numbers may be fixed in the first position.)

Furthermore, each permutation may be expressed as a product of transpositions (in many ways). For an n–cycle, this requires a minimum of $(n-1)$ transpositions. For example, $(1\;5\;3\;4) = (1\;4)(1\;3)(1\;5)$.

The permutation group on 6 symbols may be split into classes corresponding to the 11 partitions of the integer 6, as shown in the table below. The number of permutations in each class is indicated, along with the minimum number of transpositions required to express any element in the class as a product of transpositions. The total number of permutations for which the minimum number of transpositions is exactly four is $144+90+40 = 274$.

	Number of Transpositions	Number of Permutations		
one 6–cycle	5	$5!$	$=$	120
one 5–cycle, one 1–cycle	4	$6 \times 4!$	$=$	144
one 4–cycle, one 2–cycle	4	$15 \times 3!$	$=$	90
one 4–cycle, two 1–cycles	3	$15 \times 3!$	$=$	90
two 3–cycles	4	$10 \times 2! \times 2!$	$=$	40
one 3–cycle, two 2–cycle, one 1–cycle	3	$60 \times 2!$	$=$	120
one 3–cycle, three 1–cycles	2	$20 \times 2!$	$=$	40
three 2–cycles	3	15	$=$	15
two 2–cycles, two 1–cycles	2	45	$=$	45
one 2–cycle, four 1–cycles	1	15	$=$	15
six 1–cycles	0	1	$=$	1
		Total	$=$	720

[*Ed.* It is unclear in the problem statement whether we should be requiring the *minimum* number of transpositions to be equal to 4. Perhaps we should allow any transformation that can be obtained by exactly four transpositions, whether this is the minimum number or not. In this case, we should include those classes in the table for which the minimum number of transpositions is 0 or 2. (We can always apply the same transposition twice in succession with no net effect.) The answer is then $274 + 40 + 45 + 1 = 360$.]

To complete this number of the *Corner*, we turn to solutions from our readers to problems of the Grosman Memorial Mathematical Olympiad 1999 given [2001 : 423–424].

1. For every 16 positive integers n, a_1, a_2, $\ldots$, a_{15} we define

$$T(n, a_1, a_2, \ldots, a_{15}) \;=\; (a_1^n + a_2^n + \cdots + a_{15}^n) a_1 a_2 \ldots a_{15} \,.$$

Find the smallest n for which $T(n, a_1, a_2, \ldots, a_{15})$ is divisible by 15 for every choice of a_1, a_2, $\ldots$, a_{15}.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution.

We will show that $n = 4$ is the smallest n with the desired property.

First suppose that $n = 4$. Let a_1, a_2, $\ldots$, a_{15} be positive integers, and let $b = T(n, a_1, a_2, \ldots, a_{15})$. For each $i = 1$, 2, $\ldots$, 15, either $a_i \equiv 0 \pmod 3$ or $a_i^4 = a_i^2 \times a_i^2 \equiv 1 \pmod 3$, by Fermat's Little Theorem. If $a_i \equiv 0 \pmod 3$ for some i, then $b \equiv 0 \pmod 3$, since a_i is a factor of b. If $a_i \not\equiv 0 \pmod 3$ for all i, then $a_1^4 + a_2^4 + \cdots + a_{15}^2 \equiv 15 \equiv 0 \pmod 3$, and again we see that $b \equiv 0 \pmod 3$.

Similarly, for each $i = 1, 2, \ldots, 15$, either $a_i \equiv 0 \pmod 5$ or $a_i^4 \equiv 1 \pmod 5$, giving us $b \equiv 0 \pmod 5$. Therefore, $b \equiv 0 \pmod{15}$. Hence, $n = 4$ has the desired property.

Now consider positive integers $n < 4$. To see that $n = 1$ does not have the desired property, note that

$$T(1,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1) = 22 \times 128 \not\equiv 0 \pmod{15} .$$

Since $1^2 \equiv 1 \pmod 5$ and $2^2 \equiv -1 \pmod 5$, it is easy to verify that

$$T(2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1) \equiv 1 \times 2^7 \not\equiv 0 \pmod 5 .$$

Then $T(2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1) \not\equiv 0 \pmod{15}$. Thus, $n = 2$ does not have the desired property.

Since $1^3 \equiv 1 \pmod 3$ and $2^3 \equiv -1 \pmod 3$, it is easy to verify that

$$T(3,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1) \equiv 1 \times 2^7 \not\equiv 0 \pmod 3 .$$

Then $T(3,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1) \not\equiv 0 \pmod{15}$. Thus, $n = 3$ does not have the desired property.

2. Find the smallest integer n for which $0 < \sqrt[4]{n} - \lfloor \sqrt[4]{n} \rfloor < 10^{-5}$.

Remark. $\lfloor x \rfloor$ denotes the integral value of x; that is, the largest integer which does not exceed x.

Solution by Christopher J. Bradley, Bristol, UK.

Let $f(n) = n^{1/4} - \lfloor n^{1/4} \rfloor$, for positive integers n. If $n = x^4$ for some positive integer x, then $f(n) = x - x = 0$. Moreover, $f(n)$ is clearly increasing on $[x^4, (x+1)^4)$. Hence, in order to get the *smallest* integer n such that $0 < f(n) < 10^{-5}$, we must choose $n = x^4 + 1$, with x as small as possible.

Now,

$$f(x^4+1) = (1+x^4)^{1/4} - x = x\left(1 + \frac{1}{x^4}\right)^{1/4} - x ,$$

which, by the Binomial Theorem, yields

$$\begin{aligned} f(x^4+1) &= x\left(1 + \frac{1}{4}\frac{1}{x^4} + \frac{1}{4}\left(\frac{-3}{4}\right)\frac{1}{2!}\frac{1}{x^8} + \cdots\right) - x \\ &= \frac{1}{4x^3} - \frac{3}{32}\frac{1}{x^7} + \cdots \end{aligned}$$

Therefore, we look for x such that $\frac{1}{4x^3} < 10^{-5}$; that is, $4x^3 > 10^5$. Since $4 \times 30^3 = 1.08 \times 10^5$, and a value of $x = 30$ makes the subsequent terms in the binomial expansion negligible compared with 10^{-5}, it follows that $x = 30$ and $n = 30^4 + 1 = 810001$.

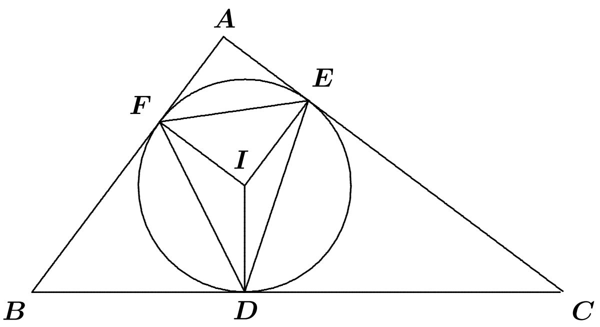
3. For every triangle ABC, denote by $D(ABC)$, the triangle whose vertices are the tangency points of the incircle of ABC (touching the sides of the triangle). The given triangle ABC is not equilateral.

(a) Prove that $D(ABC)$ is also not equilateral.

(b) Find in the sequence of triangles $T_1 = \triangle ABC$, $T_{k+1} = D(T_k)$, $k = 1, 2, \ldots$ a triangle whose largest angle α satisfies the inequality $0 < \alpha - 60^\circ < 0.0001$.

Solution by Christopher J. Bradley, Bristol, UK, modified by the editor.

(a) Let I be the incentre of $\triangle ABC$, and let D, E, F be the points of tangency of the incircle opposite A, B, C, respectively, as in the diagram below. Thus, $D(ABC) = \triangle DEF$.



Since $\angle IFA = \angle IEA = 90^\circ$, we have $\angle FIE = 180^\circ - A = B + C$ and $\angle FDE = \frac{1}{2}\angle FIE = \frac{1}{2}(B + C)$. Similarly, $\angle DEF = \frac{1}{2}(A + C)$ and $\angle EFD = \frac{1}{2}(A + B)$.

If $\triangle DEF$ is equilateral, then $A + B = A + C = B + C = 120^\circ$, from which it follows that $A = B = C = 60^\circ$, and $\triangle ABC$ is equilateral. Hence, since $\triangle ABC$ is not equilateral, neither is $\triangle DEF$.

(b) For each $k = 1, 2, \ldots$, let the vertices of T_k be A_k, B_k, C_k, and let $\alpha_k = \angle A_k$, $\beta_k = \angle B_k$, and $\gamma_k = \angle C_k$. We may suppose, without loss of generality, that the vertices are labelled so that $\alpha_k \geq \beta_k \geq \gamma_k$. Then, from the proof of part (a), we see that

$$\alpha_{k+1} = \tfrac{1}{2}(\alpha_k + \beta_k)\,, \qquad \beta_{k+1} = \tfrac{1}{2}(\alpha_k + \gamma_k)\,, \qquad \gamma_{k+1} = \tfrac{1}{2}(\beta_k + \gamma_k)\,,$$

for $k = 1, 2, \ldots$. By solving this system of difference equations, we obtain for $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
\alpha_{2n+1} &= \frac{1}{3}\left(1 + \frac{1}{2^{2n-1}}\right)\alpha_1 + \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right)(\beta_1 + \gamma_1)\,, \\
\beta_{2n+1} &= \frac{1}{3}\left(1 + \frac{1}{2^{2n-1}}\right)\beta_1 + \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right)(\alpha_1 + \gamma_1)\,, \\
\gamma_{2n+1} &= \frac{1}{3}\left(1 + \frac{1}{2^{2n-1}}\right)\gamma_1 + \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right)(\alpha_1 + \beta_1)\,.
\end{aligned}$$

We can then simplify the largest angle:

$$\begin{aligned}\alpha_{2n+1} &= \frac{1}{3}\left(1+\frac{1}{2^{2n-1}}\right)\alpha_1 + \frac{1}{3}\left(1-\frac{1}{2^{2n}}\right)(180^\circ - \alpha_1) \\ &= \left(1-\frac{1}{2^{2n}}\right)60^\circ + \left(\frac{1}{2^{2n}}\right)\alpha_1 \\ &= 60^\circ + \frac{1}{2^{2n}}(\alpha_1 - 60^\circ)\,.\end{aligned}$$

Since $60^\circ < \alpha_1 < 180^\circ$, we have $60^\circ < \alpha_{2n+1} < 60^\circ + \frac{1}{2^{2n}}120^\circ$. To obtain $0 < \alpha_{2n+1} - 60^\circ < 0.0001$, it is sufficient to choose n so that $120/2^{2n} < 0.0001$; that is, $2^{2n} > 1200000$. The smallest such n is 11, since $2^{20} \approx 1024^2 \approx 1000000$. Hence, the largest angle of triangle T_{23} satisfies the given inequality.

4. Consider a polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ with integer coefficients a, b, c, d. Prove that if $f(x)$ has exactly one real root then $f(x)$ can be factored into terms with rational coefficients.

Solution by Christopher J. Bradley, Bristol, UK..

Since the coefficients are real, complex roots occur in conjugate pairs. Hence, if there is only one real root it must be either of multiplicity 2 or of multiplicity 4. If it is of multiplicity 4, then $f(x)$ must factor as $(x-k)^4$, where $a = 4k$, $b = 6k^2$, $c = 4k^3$, and $d = k^4$; thus, $k = a/4$ is rational (and must be an integer, since k^4 is an integer).

If x_0 is a real root of multiplicity 2, then $f(x_0) = 0$ and $f'(x_0) = 0$, and we have

$$x_0^4 + ax_0^3 + bx_0^2 + cx_0 + d = 0\,, \qquad (1)$$

$$4x_0^3 + 3ax_0^2 + 2bx_0 + c = 0\,. \qquad (2)$$

Subtracting x_0 times equation (2) from 4 times equation (1) gives

$$ax_0^3 + 2bx_0^2 + 3cx_0 + 4d = 0\,. \qquad (3)$$

Subtracting 4 times equation (3) from a times equation (2) gives

$$(3a^2 - 8b)x_0^2 + 2(ab - 6c)x_0 + (ac - 16d) = 0\,. \qquad (4)$$

Subtracting $4x_0$ times equation (4) from $(3a^2 - 8b)$ times equation (2) gives

$$(9a^3 - 32ab + 48c)x_0^2 + 2(3ba^2 - 8b^2 - 2ac + 32d)x_0\,, c(3a^2 - 8b) = 0\,. \qquad (5)$$

Now x_0^2 can be eliminated from (4) and (5) to deduce the value of x_0, which we perceive to be rational.

We now quote the result, which is well known, that if a, b, c, d are integers, then any rational root of the polynomial $x^4 + ax^3 + bx^2 + cx + d = 0$ is integral. We conclude that x_0 is an integer.

Now

$$x^4 + ax^3 + bx^2 + cx + d \;=\; (x - x_0)^2 q(x)\,, \tag{6}$$

where $q(x)$ is quadratic. This quadratic is irreducible over the reals, since the roots of $q(x) = 0$ are complex. Furthermore, from (6), since x_0 is integral, $q(x)$ is of the form $x^2 + Ax + B$, where A and B are rational. In fact, since $A - 2x_0 = a$ and $B = 2x_0A + b - x_0^2$, we see that A and B are integers.

[*Ed.* The reduction process that produced equations (3) to (5) is essentially the Euclidean Algorithm being used to find the greatest common divisor of the polynomials $f(x)$ and $f'(x)$. Since both $f(x)$ and $f'(x)$ have rational coefficients, each new polynomial generated by the algorithm also has rational coefficients. Thus, even though we may look for the greatest common divisor in $\mathbb{R}[x]$, we end up with an element of $\mathbb{Q}[x]$. We know that the greatest common divisor of $f(x)$ and $f'(x)$ in $\mathbb{R}[x]$ is $x - x_0$. We conclude that $x - x_0 \in \mathbb{Q}[x]$; that is, x_0 is rational. This argument can be applied more generally to show that whenever a polynomial $f(x) \in \mathbb{Q}[x]$ has a single repeated root in any extension field of $\mathbb{Q}$ (for example, $\mathbb{R}$ or $\mathbb{C}$), that root must in fact be rational.]

5. An infinite sequence of distinct real numbers is given. Prove that it contains a subsequence of 1999 terms which is either monotonically increasing or monotonically decreasing.

Remark. The sequence of numbers a_1, $\ldots$, a_n is said to be monotone increasing if $a_1 < a_2 < \cdots < a_n$ and monotone decreasing if $a_1 > a_2 > \cdots > a_n$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solution.

Let $\{a(n)\}$ be the given sequence and let

$$S \;=\; \{m \in \mathbb{N} \mid a(m) > a(n) \text{ for all } n > m\}\,.$$

Two mutually exclusive cases can occur:

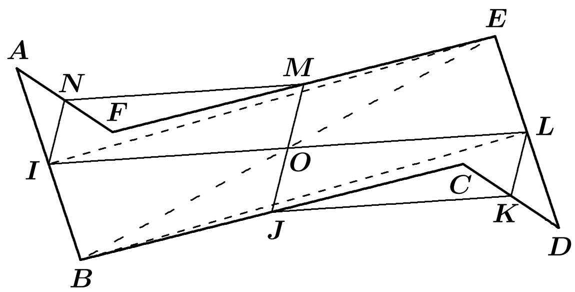
Case (i). S is infinite. Then we can certainly choose m_1, m_2, $\ldots$, m_{1999} in S such that $m_1 < m_2 < \cdots < m_{1999}$. By definition of S, the subsequence $\{a(m_j)\}_{1 \le j \le 1999}$ is decreasing.

Case (ii). S is finite. Then there exists $N_1 \in \mathbb{N}$ such that $N_1 \notin S$. Since $N_1 \notin S$, there exists $N_2 \in \mathbb{N}$ such that $N_2 > N_1$ and $a(N_1) \le a(N_2)$; we even have $a(N_1) < a(N_2)$, since the terms of the sequence are distinct. Iterating the process, we determine $N_1 < N_2 < \cdots < N_{1999}$ such that $a(N_1) < a(N_2) < \cdots < a(N_{1999})$. The subsequence $\{a(N_j)\}_{1 \le j \le 1999}$ is increasing.

6. Six points A, B, C, D, E, F are given in space. The quadrilaterals $ABDE$, $BCEF$, $CDFA$ are parallelograms. Prove that the six mid-points of the sides AB, BC, CD, DE, EF, FA are coplanar.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We use Bataille's solution.

Let I, J, K, L, M, N be the mid-points of AB, BC, CD, DE, EF, FA, respectively, and let $\mathcal{P}$ and $\mathcal{Q}$ be the planes determined by M, N, I and J, K, L, respectively. We are required to prove that $\mathcal{P} = \mathcal{Q}$. It suffices to show that the lines IL and JM are both contained in $\mathcal{P}$ and $\mathcal{Q}$ and are concurrent.



Let O be the mid-point of BE. Then, $IOMN$ is the Varignon Parallelogram of $ABEF$; hence, $O \in \mathcal{P}$. Similarly, $O \in \mathcal{Q}$ [using $JKLO$]. Since $\overrightarrow{IB} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{ED} = \overrightarrow{EL}$, we see that $IBLE$ is a parallelogram, and therefore O is the mid-point of IL. Thus, the line IL is contained in $\mathcal{P}$ (since O and I are in $\mathcal{P}$) and in $\mathcal{Q}$ (since O and L are in $\mathcal{Q}$). In a similar way, we see that O is the mid-point of JM. It follows that JM is contained in both $\mathcal{P}$ and $\mathcal{Q}$. To complete the proof, we observe that IL and JM clearly concur at O.

That completes the Olympiad Corner for this issue. Send me your contests as well as your nice solutions and generalizations.