SKOLIAD No. 76

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Please send your solutions to the problems from this issue by *1 September, 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will print solutions to problems marked with an asterisk (*) only if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

This month's problems are drawn from the county-wide and national mathematics competitions held by the Croatian Mathematical Society in 2003. Thanks to Željko Hanjš of the Croatian Mathematical Society for making these problems available.

Croatian Mathematical Society County-Wide Competition
Junior Level (Grade 1), April 4, 2003

1. The lengths of the sides of a triangle ABC are $a = b - \frac{r}{4}$, b, $c = b - \frac{r}{4}$, where r is the radius of the inscribed circle. Determine the lengths of the sides of this triangle as a function of r only.

2. If $a > 0$, determine which points (x, y) in the Cartesian plane satisfy the inequality

$$\big||x+a| - |y-a|\big| \;<\; a\,.$$

3. Find all integer solutions to the equation

$$4x + y + 4\sqrt{xy} - 28\sqrt{x} - 14\sqrt{y} + 48 \;=\; 0\,.$$

4. How many four-digit positive integers divisible by 7 have the property that, when the first and last digits are interchanged, the result is a (not necessarily four-digit) positive integer divisible by 7?

Croatian Mathematical Society National Competition Junior Level (Grade 1), May 7-10, 2003

1. Consider a triangle ABC whose sides have lengths which are prime numbers. Prove that the area of the triangle cannot be an integer.

2. The product of the positive real numbers x, y, and z is equal to 1. If

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;\geq\; x+y+z\,,$$

prove that

$$\frac{1}{x^k}+\frac{1}{y^k}+\frac{1}{z^k} \;\geq\; x^k+y^k+z^k$$

for every positive integer k.

3. Consider an isosceles triangle ABC with base length a whose two equal sides are of length b and whose altitude is of length v. If $\frac{a}{2}+v \geq b\sqrt{2}$, determine the angles of the triangle. Furthermore, if $b=8\sqrt{2}$, calculate the area of the triangle.

4. How many divisors of the number 30^{2003} are not divisors of 20^{2000}?

Next, we give the official solutions to those problems **not** marked with an asterisk $(*)$ from the Ninteenth W.J. Blundon Mathematics Contest that appeared [2003 : 261–262].

4. Find all positive numbers x such that $x^{x\sqrt{x}}=(x\sqrt{x}\,)^x$.

Solution.

Note that $(x\sqrt{x}\,)^x=\left(x^{\frac{3}{2}}\right)^x=x^{\frac{3}{2}x}$. Thus, the equation to be solved is

$$x^{x\sqrt{x}} \;=\; x^{\frac{3}{2}x}\,.$$

The equation is obviously satisfied if $x=1$. If $x\neq 1$, then we must have

$$\begin{aligned} x\sqrt{x} &= \tfrac{3}{2}x\,,\\ 2x\sqrt{x} &= 3x\,,\\ 4x^3 &= 9x^2\,,\\ 4x^3-9x^2 &= 0\,,\\ x^2(4x-9) &= 0\,,\\ x &= \tfrac{9}{4}\,, \end{aligned}$$

since $x\neq 0$ because $x>0$. Thus, the positive solutions are 1 and $\frac{9}{4}$.

5. Rationalize the denominator: $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}}$.

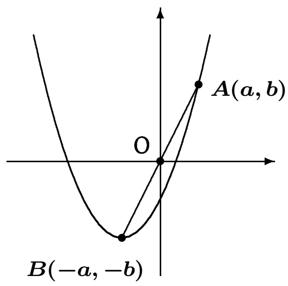
Solution.

$$
\begin{aligned}
\frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}} &= \frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}} \cdot \frac{(\sqrt{2}+\sqrt{3})-\sqrt{6}}{(\sqrt{2}+\sqrt{3})-\sqrt{6}} \\
&= \frac{\sqrt{2}+\sqrt{3}-\sqrt{6}}{5+2\sqrt{6}-6} = \frac{\sqrt{2}+\sqrt{3}-\sqrt{6}}{2\sqrt{6}-1} \\
&= \frac{\sqrt{2}+\sqrt{3}-\sqrt{6}}{2\sqrt{6}-1} \cdot \frac{2\sqrt{6}+1}{2\sqrt{6}+1} \\
&= \frac{7\sqrt{2}+5\sqrt{3}-\sqrt{6}-12}{23}.
\end{aligned}
$$

6. Points A and B are on the parabola $y = 2x^2+4x-2$. The origin is the mid-point of the line segment joining A and B. Find the length of this line segment.

Solution.

Let A have coordinates (a,b). Then, since the origin is the mid-point of the line segment AB, point B has coordinates $(-a,-b)$. Since these points are on the parabola, we must have



$$b = 2a^2+4a-2,$$

and

$$-b = 2(-a)^2+4(-a)-2 = 2a^2-4a-2.$$

Adding these two equations yields $4a^2-4=0$; that is, $a=\pm 1$. If $a=1$, then

$$b = 2(1)^2+4(1)-2 = 4.$$

If $a=-1$, then

$$b = 2(-1)^2+4(-1)-2 = -4.$$

Thus, AB is the line segment joining $(1,4)$ and $(-1,-4)$, which has length

$$\sqrt{[1-(-1)]^2+[4-(-4)]^2} = \sqrt{4+64} = \sqrt{68} = 2\sqrt{17}.$$

7. If $\log_{125} 2 = a$ and $\log_9 25 = b$, find $\log_8 9$ in terms of a and b.

Solution.

We use $\log_8 9 = \dfrac{\ln 9}{\ln 8}$. Since $\log_9 25 = b$, we have

$$\begin{aligned} \frac{\ln 25}{\ln 9} &= b\,, \\ \ln 9 &= \frac{\ln 25}{b} = \frac{2\ln 5}{b}\,. \end{aligned}$$

Since $\log_{125} 2 = a$, we have

$$\begin{aligned} \frac{\ln 2}{\ln 125} &= a\,, \\ \ln 2 &= a\ln 125 = 3a\ln 5\,, \\ \ln 8 &= 3\ln 2 = 9a\ln 5\,. \end{aligned}$$

Therefore,

$$\log_8 9 = \frac{\ln 9}{\ln 8} = \frac{(2\ln 5)/b}{9a\ln 5} = \frac{2}{9ab}\,.$$

8. Point P lies in the first quadrant on the line $y = 2x$. Point Q is a point on the line $y = 3x$ such that PQ has length 5 and is perpendicular to the line $y = 2x$. Find the point P.

Solution.

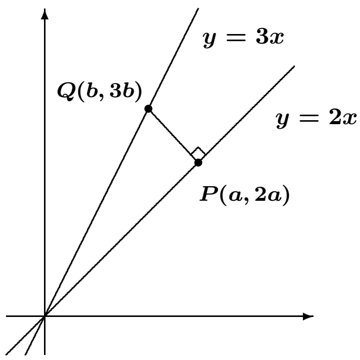
Let the coordinates of P be $(a, 2a)$ and the coordinates of Q be $(b, 3b)$. The slope of the line $y = 2x$ is 2. Hence, the slope of PQ is $-\frac{1}{2}$. Thus, we must have

$$\frac{3b-2a}{b-a} = -\frac{1}{2}\,.$$

Therefore, $b = \frac{5}{7}a$. Since $PQ = 5$, we have

$$\begin{aligned} (b-a)^2 + (3b-2a)^2 &= 5^2\,, \\ \left(-\tfrac{2}{7}a\right)^2 + \left(\tfrac{1}{7}a\right)^2 &= 25\,, \\ \tfrac{5}{49}a^2 &= 25\,, \\ a &= 7\sqrt{5}\,. \end{aligned}$$

Thus, P has coordinates $(7\sqrt{5}, 14\sqrt{5})$.



9. For what conditions on a and b is the line $x + y = a$ tangent to the circle $x^2 + y^2 = b$?

Solution.

The line is tangent to the circle if and only if the system

$$\begin{aligned} x + y &= a \\ x^2 + y^2 &= b \end{aligned}$$

has a unique solution. Solving, we get

$$\begin{aligned} x^2 + (a - x)^2 &= b\,, \\ \text{or} \quad 2x^2 - 2ax + a^2 - b &= 0\,. \end{aligned}$$

This equation has a unique solution if and only if the discriminant is zero. That is,

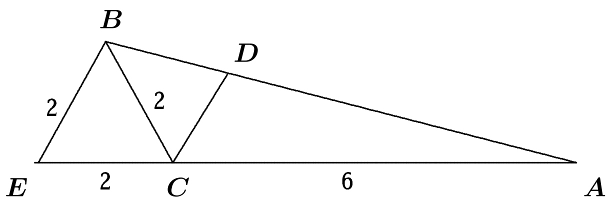
$$\begin{aligned} 4a^2 - 8(a^2 - b) &= 0\,, \\ 8b - 4a^2 &= 0\,, \\ 4(2b - a^2) &= 0\,, \\ a^2 &= 2b\,. \end{aligned}$$

Thus, the line is tangent to the circle when $a^2 = 2b$.

10. In $\triangle ABC$, we have $\angle ACB = 120$ degrees, $AC = 6$ and $BC = 2$. The internal bisector of $\angle ACB$ meets the side AB at the point D. Determine the length of the line segment CD.

Solution.

Draw BE parallel to DC meeting AC (extended) at E.



Then $\angle ACD = \angle AEB = 60°$ and $\angle DCB = \angle EBC = 60°$. Therefore, $\triangle BCE$ is an equilateral triangle. Also, $\triangle ADC$ is similar to $\triangle ABE$, since $BE \parallel CD$. Hence,

$$\begin{aligned} \frac{CD}{CA} &= \frac{EB}{EA}\,, \\ \frac{CD}{6} &= \frac{2}{8}\,, \\ CD &= \frac{3}{2}\,. \end{aligned}$$