SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2814. [2003 : 110] *Proposed by Juan José Egozcue and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be positive real numbers such that $a+b+c=abc$. Find the minimum value of

$$\sqrt{1+\frac{1}{a^2}}+\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}}\,.$$

I. *Solution by D. Kipp Johnson, Beaverton, OR, USA.*

If a, b, c are positive real numbers with $a+b+c=abc$, then we may write $a=\tan x$, $b=\tan y$, $c=\tan z$, for positive real numbers x, y, z satisfying $x+y+z=\pi$. [*Ed.* compare problem 2524 [2001 : 157].] (Thus, x, y, z are the angles of some triangle.) Making this substitution, we get

$$\sqrt{1+\frac{1}{a^2}}+\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}} \;=\; \csc x+\csc y+\csc z\,.$$

But the function $f(x)=\csc x$ is convex on the interval $(0,\pi)$. (Note that $f''(x)=\dfrac{1+\cos^2 x}{\sin^3 x}>0$ for $0<x<\pi$.) Therefore, we may apply Jensen's Inequality to obtain

$$\csc x+\csc y+\csc z \;\ge\; 3\cdot\csc\left(\frac{x+y+z}{3}\right) \;=\; 3\csc\left(\frac{\pi}{3}\right) \;=\; 2\sqrt{3}\,.$$

The value of $2\sqrt{3}$ is actually attained when $x=y=z=\frac{\pi}{3}$; that is, when $a=b=c=\sqrt{3}$. Thus, the minimum value of our expression is $2\sqrt{3}$.

II. *Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina.*

Letting $S(a,b,c)$ denote the expression that is to be minimized, we notice that

$$S(a,b,c) \;=\; \left|1+\frac{i}{a}\right|+\left|1+\frac{i}{b}\right|+\left|1+\frac{i}{c}\right|\,.$$

Applying the Triangle Inequality, we get

$$S(a,b,c) \;\ge\; \left|3+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)i\right| \;=\; \sqrt{9+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}\,.$$

The given constraint $a+b+c=abc$ is equivalent to $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=1$. Hence,

$$\begin{aligned}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 &= 2+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\\ &= 3+\frac{1}{2}\left(\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{a}-\frac{1}{b}\right)^2\right) \geq 3\,.\end{aligned}$$

Using this inequality above gives $S(a,b,c) \geq \sqrt{12} = 2\sqrt{3}$. Thus, the minimum value of $S(a,b,c)$ is $S(\sqrt{3},\sqrt{3},\sqrt{3}) = 2\sqrt{3}$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHRISTOPHER J. BRADLEY, Bristol, UK; JACQUES CHONÉ, Nancy, France; CON AMORE PROBLEM GROUP, The Danish University of Education Copenhagen, Denmark; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina (another solution); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KURT KNUEVEN, student, Northern Kentucky University, KY, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA (2 solutions); ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposers. Six solutions were either incorrect or incomplete.

Most solutions were similar to Solution I above. Some used a different initial substitution or no substitution at all, but most of these still used Jensen's Inequality. Several solutions proceeded by the method of Lagrange multipliers. These succeeded in finding the correct critical point, but failed to prove that this point corresponded to a global minimum subject to the given constraint. They were judged to be incomplete.

Murty observes that the numbers a, b, c need not be positive. This can be easily seen from Solution II, which remains valid as long as a, b, c are non-zero.

2815. [2003 : 111] Corrected. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that $\Gamma(O,R)$ is the circumcircle of $\triangle ABC$, where $\angle ACB \neq 60^\circ$. Suppose that side AB is fixed and that C varies on Γ (always on the same side of AB).

Construct equilateral triangles BCD and ACE such that A and D are on opposite sides of BC, and B and E are on opposite sides of AC.

(a) Show that CD and CE intersect Γ at fixed points F and G, respectively. Characterize these points.

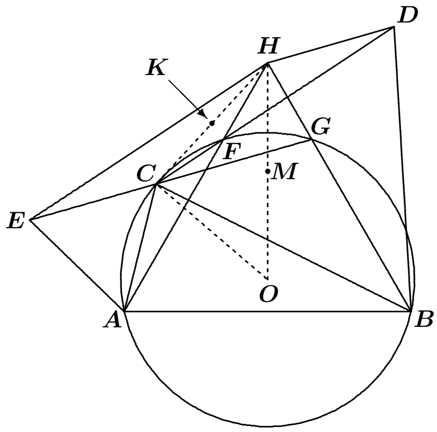
(b) Complete the parallelogram $DCEH$. Show that H is a fixed point. Characterize H.

(c) If K is the point of intersection of CH and DE, determine the locus of K as C varies.

[*Ed.* The problem as stated above incorporates a correction in part (c). Our featured solver made the correction and solved the corrected version.]

Solution by Titu Zvonaru, Bucharest, Romania.

Editor's comment. The solution makes use of directed angles. The symbol $\measuredangle XYZ$ represents the angle from line YX to line YZ. Alternatively, one can use undirected angles and analyze the cases that arise. The case presented below is valid for undirected angles when there is an acute angle at C, and the points lie on a circle in the order A, B, G, F, C.



(a) We first note that $\measuredangle BAF = \measuredangle BCF = \measuredangle BCD = 60^\circ$. We can also check that $\measuredangle GBA = \measuredangle ECA = 60^\circ$. Therefore, the points F and G are fixed: they are points on the circle Γ on the same side of AB as C such that the directed angles from BA to FA and from GB to AB are both 60°.

(b) We first show that $\triangle HDB \cong \triangle ACB$. We have $HD = EC = AC$ and $BD = BC$. Moreover,

$$\begin{aligned} \measuredangle HDB &= \measuredangle HDC + \measuredangle CDB = (180^\circ - \measuredangle DCE) + 60^\circ \\ &= 240^\circ - (360^\circ - \measuredangle ECA - \measuredangle ACB - \measuredangle BCD) \\ &= 240^\circ - (360^\circ - 60^\circ - 60^\circ - \measuredangle ACB) = \measuredangle ACB\,. \end{aligned}$$

Thus, by SAS, the triangles HDB and ACB are congruent. It follows that $HB = AB$. In a similar way, we obtain $HA = AB$. Therefore, H is fixed as the apex of the equilateral triangle with side AB located on the same side of AB as C.

(c) Let M be the mid-point of OH. Since K is the mid-point of CH, we have $MK = CO/2 = R/2$. Hence, the locus of K (as C moves on the arc of Γ from B to A) is the arc of the circle with centre M and radius $R/2$ between the mid-points of HB and HA.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2816. [2003 : 111] *Proposed by Boris Harizanov, student, Stara Zagora, Bulgaria.*

In acute-angled isosceles triangles $A_1B_1C_1$ (with $A_1C_1 = B_1C_1$) and $A_2B_2C_2$ (with $A_2C_2 = B_2C_2$), we have $A_1C_1 = A_2C_2$. For $k = 1, 2$, we have a circle with centre I_k and radius r_k inscribed in $\triangle A_kB_kC_k$, and a circle with centre O_k and radius R_k circumscribed around $\triangle A_kB_kC_k$.

If $I_1O_1 = I_2O_2$, is it true that $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ must be congruent?

Solution by Christopher J. Bradley, Bristol, UK.

The answer is NO.

In a triangle with sides a, a and c, the circumradius R and the inradius r are given by $R = \frac{a^2}{\sqrt{4a^2 - c^2}}$ and $r = \frac{c}{2}\sqrt{\frac{2a-c}{2a+c}}$. Thus,

$$OI^2 = R^2 - 2Rr = \frac{a^2(a-c)^2}{4a^2 - c^2}\,.$$

If we have two triangles with sides a, a, c and a, a, d, then

$$\begin{aligned} O_1I_1^2 = O_2I_2^2 &\implies \frac{(a-c)^2}{4a^2-c^2} = \frac{(a-d)^2}{4a^2-d^2} \\ &\implies c = d \quad \text{or} \quad (2c-5a)(2d-5a) = 9a^2\,. \end{aligned}$$

Thus, the triangles are not necessarily congruent.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Some solvers did not mention the case $OI = 0$ (but this editor has been kind to them). Zhou commented that the answer is, in fact, "YES" if IO means the **signed distance**.

2817. [2003 : 112] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Suppose that A, B, and C are the angles of $\triangle ABC$. Define

$$\begin{aligned} L &= 4\cos^2\left(\frac{A}{2}\right)\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right)\,; \\ M &= \left(\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right)\right) \\ &\qquad \prod_{\text{cyclic}}\left(\cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right) - \cos\left(\frac{A}{2}\right)\right)\,. \end{aligned}$$

Show that $L = M$.

I. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Firstly,

$$\begin{aligned} M &= \left(\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)+2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)-\cos^2\left(\frac{A}{2}\right)\right) \\ &\quad\cdot\left(\cos^2\left(\frac{A}{2}\right)-\cos^2\left(\frac{B}{2}\right)-\cos^2\left(\frac{C}{2}\right)+2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)\right) \\ &= 4\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right) \\ &\qquad -\left(\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)-\cos^2\left(\frac{A}{2}\right)\right)^2 . \end{aligned} \tag{1}$$

Since $\cos\left(\frac{B+C}{2}\right)=\sin\left(\frac{A}{2}\right)$, we have

$$\begin{aligned} &\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)-\cos^2\left(\frac{A}{2}\right) \\ &\qquad = \tfrac{1}{2}(\cos B+\cos C+2)-\left(1-\sin^2\left(\frac{A}{2}\right)\right) \\ &\qquad = \cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)+\sin^2\left(\frac{A}{2}\right) \\ &\qquad = \sin\left(\frac{A}{2}\right)\left(\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{B+C}{2}\right)\right) \\ &\qquad = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) . \end{aligned} \tag{2}$$

Substituting (2) into (1), we then have

$$M = 4\cos^2\left(\frac{B}{2}\right)\cos^2\left(\frac{C}{2}\right)\left(1-\sin^2\left(\frac{A}{2}\right)\right) = L .$$

II. *Solution by Arkady Alt, San Jose, CA, USA.*

Let $\alpha=\frac{\pi-A}{2}$, $\beta=\frac{\pi-B}{2}$, and $\gamma=\frac{\pi-C}{2}$. Then α, β, $\gamma>0$, and $\alpha+\beta+\gamma=\pi$. Thus, we can interpret α, β, γ as the angles of a triangle T. Without loss of generality, we may assume that T has circumradius $R=1/2$.

Let a, b, and c denote the sides of T. Then, by the Law of Sines, we have $a=\sin\alpha$, $b=\sin\beta$, and $c=\sin\gamma$. Note that

$$\cos\frac{A}{2} = \sin\left(\frac{\pi}{2}-\frac{A}{2}\right) = \sin\alpha = a .$$

Similarly, $\cos\left(\frac{B}{2}\right) = b$ and $\cos\left(\frac{C}{2}\right) = c$. Then, using the well-known formula $R=\frac{abc}{4K}$, where K denotes the area of T, we have

$$L = 4(abc)^2 = 64R^2K^2 = 16K^2 .$$

On the other hand, using Heron's Formula, we get

$$\begin{aligned} M &= (\sin\alpha+\sin\beta+\sin\gamma)\prod_{\text{cyclic}}(\sin\alpha+\sin\beta-\sin\gamma) \\ &= (a+b+c)(b+c-a)(c+a-b)(a+b-c) = 16K^2 . \end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; JOSEPH LING, University of Calgary, Calgary, AB; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2818. [2003 : 112] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that n, $k \geq 2$ are integers such that $(n+k^n, k) = 1$.

Prove that at least one of $n+k^n$ and $n\,k^{(k^n\,-\,1)}+1$ is not prime.

Solution by Michel Bataille, Rouen, France.

Let $p = n+k^n$ and $q = nk^{(k^n-1)}+1$. If p is composite, then we are done. Suppose instead that p is a prime. We are given that $(p,k) = 1$. By Fermat's Little Theorem, we have

$$q \;=\; (p-k^n)k^{p-n-1}+1 \;\equiv\; -k^{p-1}+1 \;\equiv\; 0 \pmod{p}\;.$$

Furthermore,

$$\begin{aligned} n\left(k^{(k^n-1)}-1\right) &\;>\; k^{(k^n-1)}-1 \;=\; \big(1+(k-1)\big)^{(k^n-1)}-1 \\ &\;\geq\; (k^n-1)(k-1) \;\geq\; k^n-1\,, \end{aligned}$$

and hence, $q = nk^{(k^n-1)}+1 > n+k^n = p$. It follows that q is composite.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER BOWEN, Halandri, Greece; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Bergen, Norway; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Most of the solutions are the same as the one featured above, except for the demonstration of the fact that $p < q$, which almost all solvers either took for granted or simply stated as "evident" or "clear". From the proof given above, it is obvious that the assumption "$(n+k^n, k) = 1$" is really superfluous. However, Parmenter is the only solver who explicitly pointed this out.

2819. [2003 : 112] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $f : \mathbb{R} \to \mathbb{R}$ satisfy, for all real x and y, $f\left(\frac{2x+y}{3}\right) \geq f\left(\sqrt[3]{x^2y}\right)$.

Prove that f is decreasing on $(-\infty, 0]$ and increasing on $[0,\infty)$.

Solution by Joseph Ling, University of Calgary, Calgary, AB, with minor modifications by the editor.

We prove that the only functions that have the given property are the constant functions. Indeed for any x,

$$f(x) \;=\; f\left(\frac{2(0)+(3x)}{3}\right) \;\geq\; f\left(\sqrt[3]{0^2(3x)}\right) \;=\; f(0)\,;$$

and by considering $y = \sqrt[3]{4}x$ and $z = -y/2$, we get

$$f(0) \;=\; f\left(\frac{2z+y}{3}\right) \;\geq\; f\left(\sqrt[3]{(-y/2)^2 y}\right) \;=\; f\left(\frac{y}{\sqrt[3]{4}}\right) \;=\; f(x)\,.$$

We remark that we can prove the claims in the proposed question if we just require that

$$f\left(\frac{2x+y}{3}\right) \;\geq\; f\left(\sqrt[3]{x^2y}\right)$$

for all real numbers x and y such that $xy \geq 0$. Suppose that x, $y \in [0,\infty)$ with $x < y$. Consider the continuous function $h(t) = 2t^3 - 3yt^2 + x^3$. Since $h(0) = x^3 \geq 0 > -y^3 + x^3 = h(y)$, there is some $u \in [0,y)$ such that $h(u) = 0$, by Intermediate Value Theorem. Thus, $2u^3 - 3yu^2 + x^3 = 0$, and hence, $x^3 = u^2(3y - 2u)$. Letting $v = 3y - 2u$, we have $v > 0$ and $x^3 = u^2v$. Therefore,

$$f(y) \;=\; f\left(\frac{2u+v}{3}\right) \;\geq\; f\left(\sqrt[3]{u^2v}\right) \;=\; f(x)\,.$$

This proves that f is increasing on $[0,\infty)$.

Let $g(x) = f(-x)$. For all x, y with $xy \geq 0$, we have

$$\begin{aligned} g\left(\frac{2x+y}{3}\right) &= f\left(\frac{2(-x)+(-y)}{3}\right) \;\geq\; f\left(\sqrt[3]{(-x)^2(-y)}\right) \\ &= f\left(-\sqrt[3]{x^2y}\right) \;=\; g\left(\sqrt[3]{x^2y}\right)\,. \end{aligned}$$

It follows that g is increasing on $[0,\infty)$, and thus, f is decreasing on $(-\infty,0]$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Maisons-Laffitte, France; CHRISTOPHER BOWEN, Halandri, Greece; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

Guersenzvaig also recognized that f is constant, and that the claims hold under the weaker hypothesis stated above. Other solvers who proved f is constant are Arslanagić, Bataille, Furdui, and Janous.

2820. [2003 : 113] *Proposed by Christopher J. Bradley, Bristol, UK.*

Suppose that Q is any point in the plane of $\triangle ABC$. Suppose that AQ, BQ, CQ meet BC, CA, AB at D, E, F, respectively; that L, M, N are the mid-points of BC, CA, AB, respectively; and that U, V, W are the mid-points of AQ, BQ, CQ, respectively.

It is known that a conic Σ passes through D, E, F, L, M, N, U, V, and W. Clearly, if Σ is enlarged by a factor of 2, with Q as the centre of enlargement, then the resulting conic Σ_Q passes through A, B, and C.

Suppose that P is any point on Σ_Q, and that lines through P parallel to AQ, BQ, CQ meet the sides BC, CA, AB at R, S, T, respectively.

Prove that R, S, and T are collinear.

Combination of solutions by David Loeffler, student, Trinity College, Cambridge, UK; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

We show that the collinearity follows quickly from the affine versions of two familiar Euclidean theorems. We can apply a linear transformation to the plane so that A, B, C, and Q are mapped to a triangle and its orthocentre: apply a shear parallel to BC until $AQ \perp BC$, then stretch the plane parallel to AQ until $BQ \perp AC$. (Compare the solution to 2822 and its reference [2].) Since the problem concerns only affinely invariant properties, it is sufficient to prove the result for the case where Q is the orthocentre of triangle ABC (so that AQ, BQ, CQ are altitudes). In this case the conic Σ through D, E, F, L, M, N, U, V, W is the nine-point circle of triangle ABC, and Σ_Q is the circumcircle. Note that for any point P in the plane, PR, PS, and PT are lines perpendicular to the sides (since they are assumed to be parallel to the altitudes). The feet of these perpendiculars, namely R, S, and T, are collinear (on the Simson line) if and only if P lies on the circumcircle of $\triangle ABC$. Since we are given that P lies on Σ_Q, we conclude that R, S, and T are collinear.

Also solved by MICHEL BATAILLE, Rouen, France; and the proposer.

2821. [2003 : 113] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

In triangle $\triangle ABC$, let w_a, w_b, w_c be the lengths of the interior angle bisectors, and r the inradius. Prove that

$$\frac{1}{{w_a}^2} + \frac{1}{{w_b}^2} + \frac{1}{{w_c}^2} \;\le\; \frac{1}{3r^2},$$

with equality if and only if $\triangle ABC$ is equilateral.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Without loss of generality, we can assume that $a \le b \le c$. Since $w_a = \dfrac{2\sqrt{bcs(s-a)}}{b+c}$, $w_b = \dfrac{2\sqrt{cas(s-b)}}{c+a}$, $w_c = \dfrac{2\sqrt{abs(s-c)}}{a+b}$, and $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$, the given inequality is equivalent to

$$\begin{aligned}4abc(a+b+c)^2 - 3a(b+c)^2(a+b-c)(a-b+c)&\\ - 3b(c+a)^2(b+c-a)(b-c+a)&\\ - 3c(a+b)^2(c+a-b)(c-a+b) &\ \ge\ 0\,.\end{aligned}$$

The left side above is equal to

$$\tfrac{1}{4}\Big((b-c)^2\Big[11ab(b-a) + 11ac(c-a) + 12bc(b+c) + 10abc - 4a^3\Big] + a(b+c)(3a+b+c)(2a-b-c)^2\Big)\,,$$

which is clearly non-negative. Thus, the inequality is true. Equality holds if and only if $a = b = c$.

Note that this proof does not require that a, b, and c are the sides of a triangle, as long as they are non-negative.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; TITU ZVONARU, Bucharest, Romania; and the proposer. There were also two incorrect solutions submitted.

This solution will probably leave our readers wondering if there is a general method behind the grouping and the factorization in Zhou's proof (see also his proof of Crux problem 2807). He claims that there is such a method and challenges the readers to figure it out.

The proposer has also asked the more general question: What is the set of all exponents p such that

$$\frac{1}{w_a^p} + \frac{1}{w_b^p} + \frac{1}{w_c^p} \le \frac{1}{3^{p-1}r^p}\ ?$$

2822. [2003 : 114] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Suppose that Π is a parallelogram with sides of lengths $2a$ and $2b$ and with acute interior angle α, and that F and F' are the foci of the ellipse Λ that is tangent to the four sides of Π at their mid-points.

(a) Find the major and minor semi-axes of Π in terms of a, b, and α.

(b) Find a straight-edge and compass construction for F and F'.

Solution by Michel Bataille, Rouen, France, with added explanations by the editor.

(a) Let $\frac{x^2}{u^2} + \frac{y^2}{v^2} = 1$ be the equation of Λ, where $u > v > 0$. Let T_j ($j = 1, 2, 3, 4$) be the mid-points of the sides of Π (see Figure 1).

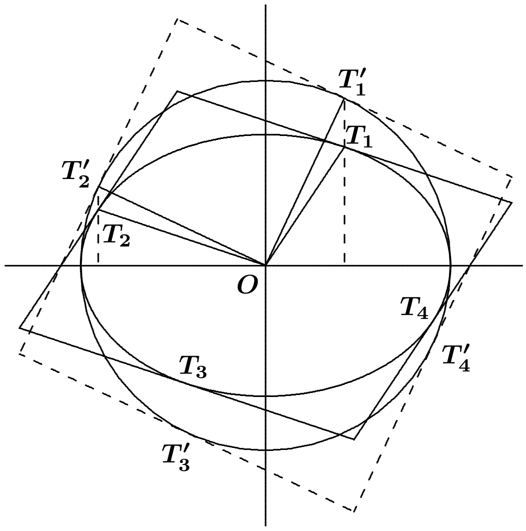


Figure 1

Consider the normal perspective affinity $\mathcal{A}$ whose axis is the x–axis and whose scale factor is u/v. (*Normal* implies that the strain is in the direction of the y–axis; see reference [2] for an elementary and pleasant treatment of perspective affinities and their link to ellipses.) The ellipse Λ is transformed by $\mathcal{A}$ into its principal circle Λ' (centre O, radius u). Since $\mathcal{A}$ preserves parallelism, mid-points, and contacts, $\mathcal{A}$ transforms Π into a parallelogram Π' whose sides touch Λ' at their mid-points $T_j' = \mathcal{A}(T_j)$ ($j = 1, 2, 3, 4$). As such, Π' is a square.

Now, let θ be a real number such that $T_1' = (u\cos\theta, u\sin\theta)$ (which implies that $T_1 = (u\cos\theta, v\sin\theta)$). Then, since $OT_1' \perp OT_2'$, we have

$$T_2 = \left(u\cos(\theta + \tfrac{\pi}{2}), v\sin(\theta + \tfrac{\pi}{2})\right) = (-u\sin\theta, v\cos\theta)\,.$$

It follows that

$$\begin{aligned} a^2 &= OT_1^2 = u^2\cos^2\theta + v^2\sin^2\theta\,, \\ b^2 &= OT_2^2 = u^2\sin^2\theta + v^2\cos^2\theta\,. \end{aligned}$$

Moreover, the area of the parallelogram with sides $\overrightarrow{OT_1}$ and $\overrightarrow{OT_2}$ is $OT_1 \cdot OT_2 \sin\alpha = ab\sin\alpha$. Since this area is a quarter the area of Π, and since, for any ellipse, all parallelograms that are tangent at their mid-points have the same area, it follows that $uv = ab\sin\alpha$. From these relations we obtain first $a^2 + b^2 = u^2 + v^2$, then $u \pm v = (a^2 + b^2 \pm 2ab\sin\alpha)^{1/2}$, and finally

$$\begin{aligned} u &= \tfrac{1}{2}\left((a^2+b^2+2ab\sin\alpha)^{1/2} + (a^2+b^2-2ab\sin\alpha)^{1/2}\right) , \\ v &= \tfrac{1}{2}\left((a^2+b^2+2ab\sin\alpha)^{1/2} - (a^2+b^2-2ab\sin\alpha)^{1/2}\right) . \end{aligned}$$

(b) Given Π (and the mid-points T_j, ($j = 1, 2, 3, 4$) of its sides), we will construct the axes and vertices of Λ. The foci are then readily obtained. First, draw the circle Γ with diameter T_2T_4, and denote by S_1, S_3 the points of intersection of Γ with the perpendicular to T_2T_4 at the centre O of Π. The circle centred on the line T_2T_4 and passing through S_1 and T_1 meets T_2T_4 at U, V. (See Figure 2.)

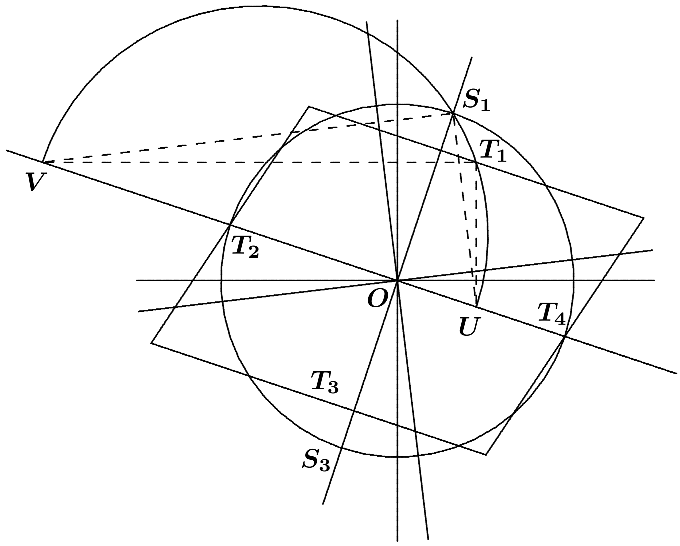


Figure 2

Let $\mathcal{B}$ be the perspective affinity with axis T_2T_4 that transforms S_1 into T_1, and S_3 into T_3. Claim: $\mathcal{B}$ transforms Γ into our desired conic Λ. [$\mathcal{B}$ takes Γ into a conic through T_1, T_2, T_3, T_4. We must show, therefore, that the sides of the given parallelogram are the images under $\mathcal{B}$ of tangents to Γ. The tangent to Γ at S_1 is taken by $\mathcal{B}$ to the line parallel to it through T_1; but that line must be the side of the parallelogram through T_1, since that side is perpendicular to OS_1, as is the tangent to Γ through S_1. As for T_2, it is a fixed point; thus, the tangent to Γ at T_2, which is parallel to OS_1, must be taken into the line through T_2 that is parallel to OT_1, which is a side of the parallelogram, as claimed.]

Note that since UV is a diameter of Γ, US_1 and VS_1 are perpendicular, as are their images UT_1 and VT_1. This provides two perpendicular directions transformed by $\mathcal{B}$ into two perpendicular directions. It follows that the axes of Λ are the lines through O that are parallel to UT_1 and VT_1. [The lines through O that are parallel to US_1 and VS_1 are perpendicular, and therefore conjugate with respect to Γ. These are taken into two lines through O that are conjugate with respect to Λ; these two lines are also perpendicular. The axes of a conic are the two lines through its centre that are both conjugate and perpendicular.] Consequently, the vertices of Λ must be the images under $\mathcal{B}$ of the points of intersection of Γ with the diameters parallel to US_1 and VS_1. (See Figure 3, where, for practical reasons, S_3 and its image T_3 have been used instead of S_1 and T_1 in constructing A, a vertex on the major axis, and B, a vertex on the minor axis.) Finally, note that the foci are the intersection points of line OA with the circle centred at B whose radius is OA [since the distance c from O to the foci satisfies $c^2 = u^2 - v^2 = OA^2 - OB^2$].

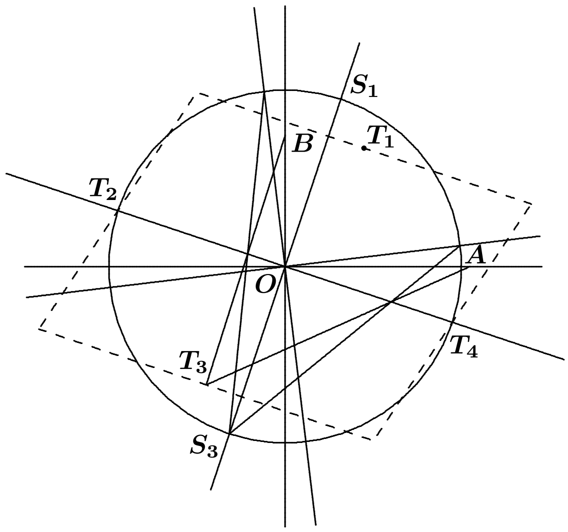


Figure 3

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (part (a) *only); LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

In [3] *Dan Pedoe shows how to construct the centre and foci of an ellipse given only five of its points. Konečný provided, instead of a construction, reference* [1] *where it is shown how to inscribe an ellipse in a given parallelogram with tangency at a prescribed point of one side. He also recommends the essay by Naoki Sato* [4].

References

[1] Heinrich Dörrie, *100 Great problems of Elementary Mathematics*. Dover, 1965. (German title: *Triumph der Mathematik*.)

[2] Max Jeger, *Transformation Geometry*. Allen and Unwin Ltd., 1966.

[3] Dan Pedoe, Pascal Redivivus II, *Crux Math.* [1979 : 281–287]

[4] Naoki Sato, Ellipses in polygons. *Crux with Mayhem* [2000 : 361–371].

2823. [2003 : 114] *Proposed by Christopher J. Bradley, Bristol, UK.*

Suppose that L, M, N are points on BC, CA, AB, respectively, and are distinct from A, B and C. Suppose further that

$$\frac{BL}{LC} = \frac{1-\lambda}{\lambda}, \quad \frac{CM}{MA} = \frac{1-\mu}{\mu}, \quad \text{and} \quad \frac{AN}{NB} = \frac{1-\nu}{\nu},$$

and that the circles AMN, BNL, and CLM meet at the Miquel point P.

Find $[BCP] : [CAP] : [ABP]$ in terms of λ, μ, ν and the side lengths of $\triangle ABC$.

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

We will solve the problem using barycentric (areal) coordinates with ABC as the triangle of reference.

The barycentric coordinates of the given points in this problem are

$$A(1,0,0), \quad B(0,1,0), \quad C(0,0,1),$$
$$L(0,\lambda,1-\lambda), \quad M(1-\mu,0,\mu), \quad N(\nu,1-\nu,0).$$

In order to solve the problem, we need the barycentric coordinates of the point P, since these are proportional to the areas $[BCP]$, $[CAP]$, and $[ABP]$.

It is known that, in barycentric coordinates (x,y,z), the equation of a circle has the form

$$a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0, \qquad (1)$$

where a, b, c are the sides of $\triangle ABC$ opposite the vertices A, B, C, respectively, and p, q, r are the powers of the points A, B, C, respectively, with respect to the circle. We note in passing that the equation of the circumcircle of ABC is simply

$$a^2yz + b^2zx + c^2xy = 0.$$

(As a reference for these results, see, for instance, the article in the digital journal FORUM GEOMETRICORUM with URL

`http://www.math.fau.edu/yiu/clawson.pdf.`

This article is about the so-called "Clawson Point" of a triangle, referring to an old problem in ***Crux Mathematicorum*** [1983 : 23–24]. The article attributes the result (1) to John Conway.)

First, we will obtain the equation of the circle CLM. We use (1) to make the following conclusions: Since the circle passes through $C(0,0,1)$, we have $r = 0$; since the circle passes through $L(0,\lambda,1-\lambda)$, we obtain $q = a^2(1-\lambda)$; since the circle passes through $M(1-\mu,0,\mu)$, we get $p = b^2\mu$. Therefore, the equation of circle CLM is

$$a^2yz + b^2zx + c^2xy - (x+y+z)\big(b^2\mu x + a^2(1-\lambda)y\big) = 0. \qquad (2)$$

Analogously, the equation of circle BNL is

$$a^2yz + b^2zx + c^2xy - (x+y+z)\big(c^2(1-\nu)x + a^2\lambda z\big) \;=\; 0\,. \qquad (3)$$

These two circles intersect at P (and at L). By solving equations (2) and (3), perhaps with the help of a computer algebra system such as MAPLE, it can be shown that the barycentric coordinates of P are

$$\begin{aligned} x &= a^2\big[-a^2\lambda(1-\lambda) + b^2(1-\lambda)(1-\mu) + c^2\lambda\mu\big]\,, \\ y &= b^2\big[a^2\lambda\mu - b^2\mu(1-\mu) + c^2(1-\mu)(1-\nu)\big]\,, \\ z &= c^2\big[a^2(1-\lambda)(1-\nu) + b^2\mu\nu - c^2\nu(1-\nu)\big]\,. \end{aligned}$$

The ratio of these coordinates is $[BCP] : [CAP] : [ABP]$.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let R, r, s, and t be the circumradii of $\triangle ABC$, $\triangle AMN$, $\triangle BNL$, and $\triangle CLM$, respectively. Let $x = MN$, $y = NL$, $z = LM$, $u = LP$, $v = MP$, $w = NP$, and $\theta = \angle AMP = \angle BNP = \angle CLP$. Applying Ptolemy's Theorem in the cyclic quadrilaterals $CLPM$, $AMPN$, and $BNPL$, we get

$$\begin{aligned} \begin{bmatrix} (1-\mu)b & \lambda a & 0 \\ 0 & (1-\nu)c & \mu b \\ \nu c & 0 & (1-\lambda)a \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} &= \begin{bmatrix} CP\cdot z \\ AP\cdot x \\ BP\cdot y \end{bmatrix} \\ &= 2\sin\theta \begin{bmatrix} tz \\ rx \\ sy \end{bmatrix}. \end{aligned}$$

By Cramer's Rule, the ratio $u : v : w$ is given by $D_1 : D_2 : D_3$, where

$$D_1 = \begin{vmatrix} tz & \lambda a & 0 \\ rx & (1-\nu)c & \mu b \\ sy & 0 & (1-\lambda)a \end{vmatrix}, \quad D_2 = \begin{vmatrix} (1-\mu)b & tz & 0 \\ 0 & rx & \mu b \\ \nu c & sy & (1-\lambda)a \end{vmatrix},$$

$$\text{and } D_3 = \begin{vmatrix} (1-\mu)b & \lambda a & tz \\ 0 & (1-\nu)c & rx \\ \nu c & 0 & sy \end{vmatrix}.$$

By the Sine Law, $\frac{z}{t} = 2\sin C = \frac{c}{R}$. Thus, $\frac{tz}{R} = \frac{z^2}{c} = n$, where

$$\begin{aligned} n &= \frac{\lambda^2a^2 + (1-\mu)^2b^2 - 2\lambda(1-\mu)ab\cos C}{c} \\ &= \frac{\lambda^2a^2 + (1-\mu)^2b^2 - \lambda(1-\mu)(a^2+b^2-c^2)}{c}\,, \end{aligned}$$

by the Cosine Law. Likewise, we have

$$\begin{aligned} \frac{rx}{R} &= \ell = \frac{\mu^2b^2 + (1-\nu)^2c^2 - \mu(1-\nu)(b^2+c^2-a^2)}{a}\,, \\ \frac{sy}{R} &= m = \frac{\nu^2c^2 + (1-\lambda^2)a^2 - \nu(1-\lambda)(c^2+a^2-b^2)}{b}\,. \end{aligned}$$

Hence, $[BCP] : [CAP] : [ABP] = au : bv : cw = T_1 : T_2 : T_3$, where

$$\begin{aligned} T_1 &= a\begin{vmatrix} n & \lambda a & 0 \\ \ell & (1-\nu)c & \mu b \\ m & 0 & (1-\lambda)a \end{vmatrix}, \\ T_2 &= b\begin{vmatrix} (1-\mu)b & n & 0 \\ 0 & \ell & \mu b \\ \nu c & m & (1-\lambda)a \end{vmatrix}, \\ T_3 &= c\begin{vmatrix} (1-\mu)b & \lambda a & n \\ 0 & (1-\nu)c & \ell \\ \nu c & 0 & m \end{vmatrix}. \end{aligned}$$

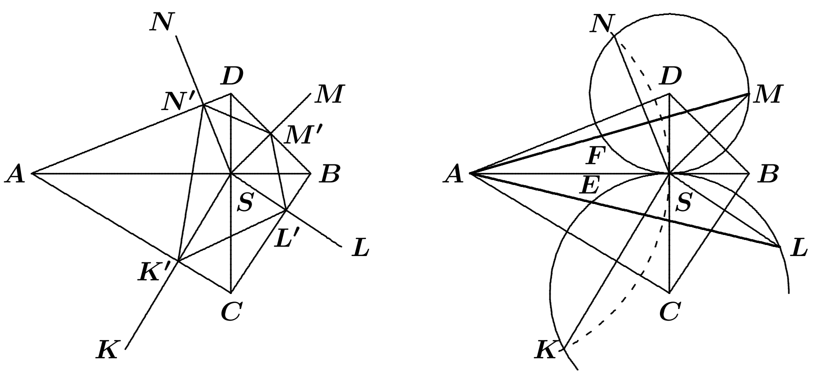
Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

2824. [2003 : 115] *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Two perpendicular line segments AB and CD intersect at S. Denote by K, L, M, N the reflections of S in the lines AC, BC, BD, AD, respectively. Suppose that the circumcircle of $\triangle SKL$ meets the line AL again in E, and that the circumcircle of $\triangle SMN$ meets the line AM again in F.

Prove that quadrilateral $KEFN$ is cyclic.

Solution by John G. Heuver, Grande Prairie, AB.



Let SK, SL, SM, SN intersect AC, BC, BD, AD in K', L', M', N', as in the diagram above. [Because of the right angles at K' and N', SA is a diameter of circle $SK'AN'$, with analogous statements for the circles having diameters SB, SC, and SD.] It follows, as shown in the diagram, that

$$\begin{aligned} \angle L'K'N' + \angle N'M'L' &= \angle L'K'S + \angle SK'N' + \angle N'M'S + \angle SM'L' \\ &= \angle BCS + \angle SAD + \angle ADS + \angle SBC \;=\; \pi\,. \end{aligned}$$

This implies that quadrilateral $K'L'M'N'$ is cyclic; hence, quadrilateral $KLMN$ is cyclic, since its sides are parallel to the former. Further, because AD and BD are perpendicular bisectors of SN and SM, it follows that D is the center of circle SNM. Similarly, C is the center of circle SKL, from which follows that AB is tangent to both circles. We have $AS = AK = AN$. The circle with centre A and radius AS inverts circles SNM and SKL into themselves, since they are orthogonal to the circle of inversion. Points L and M invert into E and F, respectively, while points K and N are invariant. Thus the circumcircle of $KLMN$ is inverted into the circumcircle of quadrilateral $KEFN$. More precisely, the points K, E, F, N lie on a circle, or (should circle $KLMN$ contain A) they lie on a line.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

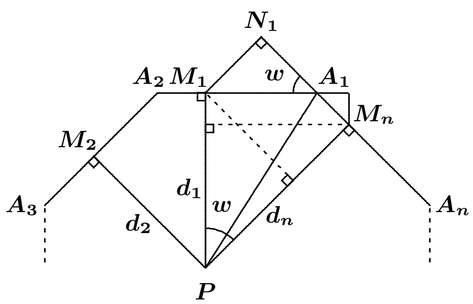
2825★. [2003 : 115] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\mathcal{R}_n$ be a regular n–gon ($n \geq 3$), and let $\mathcal{P}_n$ be the set of all points P in $\mathcal{R}_n$ such that all n perpendiculars from P to the sides of $\mathcal{R}_n$ have feet lying in the interior of the respective sides. These feet, the endpoints of the respective sides, and the point P form $2n$ (right-angled) triangles. Let S_1 and S_2 be the sums of the areas of n triangles each, using alternate triangles.

Prove that $S_1 = S_2$ for all points P in $\mathcal{P}_n$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let d_1, d_2, …, d_n be the distances PM_1, PM_2, …, PM_n, respectively, from P to the sides of the polygon (with M_i on A_iA_{i+1}, where the subscripts are taken modulo n). Let N_1 be the orthogonal projection of M_1 onto the side A_1A_n, and let $w = \angle M_1A_1N_1 = \angle M_1PM_n$ be the external angle of the regular n–gon.



We have

$$A_1M_1 \;=\; M_1N_1 \csc w \;=\; (d_n - d_1 \cos w) \csc w \;=\; d_n \csc w - d_1 \cot w \,,$$

and

$$2 \times \text{Area } (PA_1M_1) \;=\; d_1 \cdot A_1M_1 \;=\; d_nd_1 \csc w - d_1^2 \cot w\,.$$

Hence, going counterclockwise, we get

$$2S_1 \;=\; (d_nd_1 + d_1d_2 + \cdots + d_{n-1}d_n) \csc w - (d_1^2 + d_2^2 + \cdots + d_n^2) \cot w\,.$$

Similarly, by projecting M_n on the side A_1A_2, we have

$$2 \times \text{Area } (PA_1M_n) \;=\; d_n \cdot A_1M_n \;=\; d_nd_1 \csc w - d_n^2 \cot w\,,$$

and, going clockwise, we get

$$2S_2 \;=\; (d_nd_1 + d_{n-1}d_n + \cdots + d_1d_2) \csc w - (d_n^2 + d_{n-1}^2 + \cdots + d_1^2) \cot w\,.$$

Thus, $S_1 = S_2$.

Also solved by MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Our problem bears a superficial resemblance to a "proof without words" from ten years ago [“Proof Without Words: Fair Allocation of a Pizza” by Larry Carter and Stan Wagon, Math. Mag. **67***:4 (Oct. 1994) p. 267]:*

> If a pizza is divided into $2n$ slices by making cuts at angles of π/n from an arbitrary point P in the pizza, then the sums of the areas of alternate slices are equal when n is even and greater than 2. For general P the sums are not equal when n is 2 or odd.

The authors provide a partial proof and several references, including Crux problem 1325 [1989 : 120–122].

Most solvers of our problem determined the area of S_1 explicitly, showing it to be half the area of the polygon. Loeffler pointed out two worthy by-products that follow from such an approach: (1) *for any point P in a regular n–gon centred at O, the centroid of the pedal n–gon is the mid-point of OP; and* (2) $S_1 = S_2$ *for any point P in the plane, provided that we use signed areas. When Loeffler's generalization is applied to a polygonal pizza, however, even a pure mathematician might object if his portion of the pizza were to contain too much negative area.*