Some Necessary Conditions for a Real Polynomial to have only Real Roots

C.H. Harris Kwong and Amitabha Tripathi

The purpose of this note is to derive some necessary conditions for a real polynomial of degree greater than one to have all its roots real. While there are several articles [2, 3, 4, 5] that deal with related problems, our results are simple and elementary.

Theorem 1 Let $n \geq 2$, and let a_0, a_1, $\ldots$, a_n be real numbers with $a_n \neq 0$. If all roots of

$$p(x) \;=\; a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $(n-1)a_1^2 \geq 2na_0a_2$.

Proof. If $a_0 = 0$, we have nothing to prove. Hence, we may assume $a_0 \neq 0$, so that all the roots of $p(x)$ are non-zero. Since the roots of $p(x)$ are real, so are the roots of

$$q(x) \;=\; x^n p(1/x) \;=\; a_n + a_{n-1}x + \cdots + a_2 x^{n-2} + a_1 x^{n-1} + a_0 x^n\,.$$

It follows from Rolle's Theorem that all the roots of $q^{(k)}(x)$ are real for each k, where $0 \leq k \leq n-1$. In particular, the roots of

$$q^{(n-2)}(x) \;=\; (n-2)!\,a_2 + (n-1)!\,a_1 x + \frac{n!}{2}\,a_0 x^2$$

are real. This implies

$$[(n-1)!\,a_1]^2 \;\geq\; 4\cdot(n-2)!\,a_2\cdot\frac{n!}{2}\,a_0\,,$$

or equivalently, $(n-1)a_1^2 \geq 2na_0a_2$. □

The inequality in Theorem 1 is also a sufficient condition for all roots to be real if $n = 2$. However, this condition is no longer sufficient when $n \geq 3$. For instance,

$$p(x) \;=\; x(x^{n-1} + x^{n-2} + \cdots + x + 1)$$

has roots 0 and $e^{\frac{2\pi ik}{n}}$, where $1 \leq k \leq n-1$, although $(n-1)a_1^2 \geq 2na_0a_2$.

We can extend Theorem 1 to any three consecutive coefficients.

Theorem 2 Let $n \geq 2$, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_n \neq 0$. If all roots of

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $(n-k-1)(k+1)a_{k+1}^2 \geq (n-k)(k+2)a_k a_{k+2}$ for each k, where $0 \leq k \leq n-2$.

Proof. Since the roots of

$$p^{(k)}(x) = \sum_{m=k}^{n} \binom{m}{k} k!\, a_m x^{m-k}$$

are real, Theorem 1 implies that

$$(n-k-1)\left[\binom{k+1}{k} k!\, a_{k+1}\right]^2 \geq 2(n-k)\left[\binom{k}{k} k!\, a_k\right]\left[\binom{k+2}{k} k!\, a_{k+2}\right],$$

so that

$$(n-k-1)(k+1)a_{k+1}^2 \geq (n-k)(k+2)a_k a_{k+2}. \tag{1}$$

This completes the proof. □

Corollary 1 Let $n \geq 2$, and let $a_0, a_1, \ldots, a_n$ be positive real numbers. If all roots of

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $a_1 a_{n-1} \geq n^2 a_0 a_n$.

Proof. From inequality (1), we have

$$\prod_{k=0}^{n-2} \frac{k+1}{k+2} \prod_{k=0}^{n-2} \frac{a_{k+1}}{a_k} \geq \prod_{k=0}^{n-2} \frac{n-k}{n-k-1} \prod_{k=0}^{n-2} \frac{a_{k+2}}{a_{k+1}},$$

which reduces to

$$\frac{1}{n} \cdot \frac{a_{n-1}}{a_0} \geq \frac{n}{1} \cdot \frac{a_n}{a_1}.$$

Hence, $a_1 a_{n-1} \geq n^2 a_0 a_n$. □

Next we have an interesting proof of a slightly weaker version of Theorem 1.

Theorem 3 Let $n \geq 2$, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_n \neq 0$. If all roots of

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $na_1^2 \geq 8a_0 a_2$.

Proof. We may assume $a_0 a_2 \neq 0$, for otherwise there is nothing to prove. Thus, all roots of $p(x)$ are real and non-zero; let them be $\alpha_1, \alpha_2, \ldots, \alpha_n$. For each j, set $m_j = a_n \alpha_j^{n-2} + a_{n-1}\alpha_j^{n-3} + \cdots + a_3 \alpha_j + a_2$. Then we have

$$0 \;=\; p(\alpha_j) \;=\; m_j \alpha_j^2 + a_1 \alpha_j + a_0\,. \tag{2}$$

Thus, the real quadratic polynomial $m_j x^2 + a_1 x + a_0$ has a real root α_j. Therefore,

$$a_1^2 \;\geq\; 4 a_0 m_j \qquad \text{for} \quad 1 \leq j \leq n\,. \tag{3}$$

Summing inequality (3) over all j, we have, because of equation (2),

$$n a_1^2 \;\geq\; 4 a_0 \sum_{j=1}^{n} m_j \;=\; -4 a_0 a_1 \sum_{j=1}^{n} \alpha_j^{-1} - 4 a_0^2 \sum_{j=1}^{n} \alpha_j^{-2}\,. \tag{4}$$

Since α_j^{-1} are the roots of the polynomial $q(x) = x^n p(1/x) = \sum_{i=0}^{n} a_i x^{n-i}$, we have

$$\sum_{j=1}^{n} \alpha_j^{-1} \;=\; -\frac{a_1}{a_0}\,. \tag{5}$$

From $\sum_{j=1}^{n} \alpha_j^{-2} = \Big(\sum_{j=1}^{n} \alpha_j^{-1}\Big)^2 - 2 \sum_{1 \leq j < k \leq n} (\alpha_j \alpha_k)^{-1}$, we find

$$\sum_{j=1}^{n} \alpha_j^{-2} \;=\; \Big(-\frac{a_1}{a_0}\Big)^2 - 2\,\frac{a_2}{a_0}\,, \tag{6}$$

The proof is completed by substituting (5) and (6) into (4). □

We derive from Theorem 3, in a manner analogous to Theorem 2 and Corollary 1, the following results.

Theorem 4 Let $n \geq 2$, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_n \neq 0$. If all roots of

$$p(x) \;=\; a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $(n-k)(k+1) a_{k+1}^2 \;\geq\; 4(k+2) a_k a_{k+2}$ for each k, where $0 \leq k \leq n-2$.

Corollary 2 Let $n \geq 2$, and let $a_0, a_1, \ldots, a_n$ be positive real numbers. If all roots of

$$p(x) \;=\; a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

are real, then $(n-1)!\, a_1 a_{n-1} \geq 4^{n-1} a_0 a_n$.

References

[1] W.S. Burnside and A.W. Panton, *The Theory of Equations*, Dover, NY, 1960.

[2] Á. Elbert, Some inequalities concerning polynomials having only real roots, *Studia Sci. Math. Hungar.* **6** (1971), 251–261.

[3] P. Erdős and T. Grünwald, On polynomials with only real roots, *Ann. of Math.* **40** (1939), 537–548.

[4] J. Pitman, Probabilistic bounds on the coefficients of polynomials with only real roots, *J. Combin. Theory Ser. A* **77** (1997), 279–303.

[5] Q.I. Rahman and A.O. Watt, On trigonometric polynomials with only real zeros, *J. Math. Anal. Appl.* **182** (1994), 405–408.

C.H. Harris Kwong
Department of Mathematics
& Computer Science
SUNY College at Fredonia
Fredonia, NY, USA 14063
kwong@cs.fredonia.edu

Amitabha Tripathi
Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi
110016, India
atripath@maths.iitd.ac.in